From marked surfaces to clannish algebras

Yu Zhou
(joint work with Yu Qiu)

Universität Bielefeld

Soltau, March 26, 2014

Marked surfaces, I

- A connected orientable compact surface with empty boundary is determined by its genus up to homeomorphism.

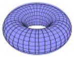
Marked surfaces, I

- A connected orientable compact surface with empty boundary is determined by its genus up to homeomorphism.
  - $g = 0$: sphere;

Marked surfaces, I

- A connected orientable compact surface with empty boundary is determined by its genus up to homeomorphism.
  - $g = 0$: sphere;

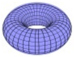
  - $g = 1$: torus;



Marked surfaces, I

- A connected orientable compact surface with empty boundary is determined by its genus up to homeomorphism.
  - $g = 0$: sphere;

  - $g = 1$: torus;



  - $g > 1$: the connected sum of g tori.

$(g = 3)$

Marked surfaces, II

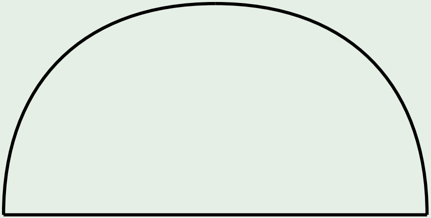
- A connected orientable compact surface with non-empty boundary is obtained from a surface above by removing some disks.

Marked surfaces, II

- A connected orientable compact surface with non-empty boundary is obtained from a surface above by removing some disks.

Example

This surface is homeomorphic to the surface obtained from a sphere by removing a disk.



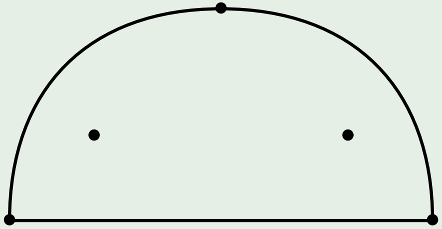
Marked surfaces, III

- A marked surface **S** is a connected oriented compact surface S with non-empty boundary and with a finite set M of marked points lying in S such that there is at least one marked point in each boundary component.

Marked surfaces, III

- A marked surface **S** is a connected oriented compact surface S with non-empty boundary and with a finite set M of marked points lying in S such that there is at least one marked point in each boundary component.

Example



Marked surfaces, IV

- A marked surface **S** is determined by the following data (up to homeomorphism):

Marked surfaces, IV

- A marked surface $\mathbf{S}$ is determined by the following data (up to homeomorphism):
  - its genus g,

Marked surfaces, IV

- A marked surface $\mathbf{S}$ is determined by the following data (up to homeomorphism):
  - its genus g,
  - the number b of its boundary components,

Marked surfaces, IV

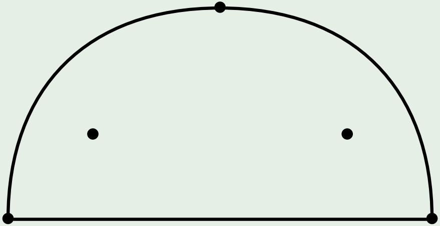
- A marked surface **S** is determined by the following data (up to homeomorphism):
  - its genus g,
  - the number b of its boundary components,
  - the numbers $m_1, \cdots, m_b$ of the marked points in each boundary components, and

Marked surfaces, IV

- A marked surface **S** is determined by the following data (up to homeomorphism):
  - its genus g,
  - the number b of its boundary components,
  - the numbers $m_1, \cdots, m_b$ of the marked points in each boundary components, and
  - the number p of punctures.

Marked surfaces, V

Example



$g = 0;\ b = 1;\ m_1 = 3;\ p = 2.$

Clannish algebras, I

- $\mathbf{k}$: an algebraic closed field

Clannish algebras, I

- $\mathbf{k}$: an algebraic closed field
- Q: a quiver

Clannish algebras, I

- $\mathbf{k}$: an algebraic closed field
- Q: a quiver
- Sp: a set of loops in Q

Clannish algebras, I

- **k**: an algebraic closed field
- Q: a quiver
- Sp: a set of loops in Q
- Z: a set of compositions $\mu\nu$ of arrows μ, ν which are not in Sp

Clannish algebras, I

- $\mathbf{k}$: an algebraic closed field
- Q: a quiver
- Sp: a set of loops in Q
- Z: a set of compositions $\mu\nu$ of arrows μ, ν which are not in Sp

Definition (Crawley-Boevey)

The algebra $A := \mathbf{k}Q/Z \cup \{e^2 - e \mid e \in Sp\}$ is called clannish if the following conditions hold:

Clannish algebras, I

- **k**: an algebraic closed field
- Q: a quiver
- Sp: a set of loops in Q
- Z: a set of compositions $\mu\nu$ of arrows μ, ν which are not in Sp

Definition (Crawley-Boevey)

The algebra $A := \mathbf{k}Q/Z \cup \{e^2 - e \mid e \in Sp\}$ is called clannish if the following conditions hold:

- at most two arrows start at each vertex, at most two stop;

Clannish algebras, I

- **k**: an algebraic closed field
- Q: a quiver
- Sp: a set of loops in Q
- Z: a set of compositions $\mu\nu$ of arrows μ, ν which are not in Sp

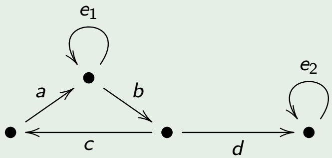
Definition (Crawley-Boevey)

The algebra $A := \mathbf{k}Q/Z \cup \{e^2 - e \mid e \in Sp\}$ is called clannish if the following conditions hold:

- at most two arrows start at each vertex, at most two stop;
- for each arrow $a \notin Sp$, there is at most one arrow b with $ba \notin Z$ and at most one arrow c with $ac \notin Z$.

Clannish algebras, II

Example



$$Sp = \{e_1, e_2\};$$

$$Z = \{ba, ac, cb\}.$$

Clannish algebras, III

- For clannish algebras, we have the following known results:

Clannish algebras, III

- For clannish algebras, we have the following known results:
  1. a classification of the indecomposable modules was given by Crawley-Boevey, Bondarenko and Deng (around 1990);

Clannish algebras, III

- For clannish algebras, we have the following known results:
  1. a classification of the indecomposable modules was given by Crawley-Boevey, Bondarenko and Deng (around 1990);
  2. a description of the homomorphism spaces between the indecomposables was given by Geiß (1999).

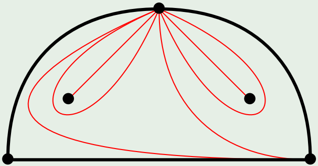
Admissible triangulations

- Admissible triangulation: a maximal collection Γ of arcs in $\mathbf{S}$ connecting marked points such that they do not cross each other and each puncture is enclosed by a loop in Γ.

Admissible triangulations

- Admissible triangulation: a maximal collection Γ of arcs in $\mathbf{S}$ connecting marked points such that they do not cross each other and each puncture is enclosed by a loop in Γ.

Example



From triangulations to clannish algebras, I

- Fix an admissible triangulation Γ of $\mathbf{S}$. We associate a quiver $Q^\Gamma = (Q_0^\Gamma, Q_1^\Gamma)$ with Sp^Γ and Z^Γ as follows:

From triangulations to clannish algebras, I

- Fix an admissible triangulation Γ of $\mathbf{S}$. We associate a quiver $Q^\Gamma = (Q_0^\Gamma, Q_1^\Gamma)$ with Sp^Γ and Z^Γ as follows:
  - Q_0^Γ: the arcs in Γ whose endpoints are in the boundary.

From triangulations to clannish algebras, I

- Fix an admissible triangulation Γ of $\mathbf{S}$. We associate a quiver $Q^\Gamma = (Q_0^\Gamma, Q_1^\Gamma)$ with Sp^Γ and Z^Γ as follows:
  - Q_0^Γ: the arcs in Γ whose endpoints are in the boundary.
  - $Q_1^\Gamma \setminus Sp^\Gamma$: the oriented angles between two edges of a triangle.

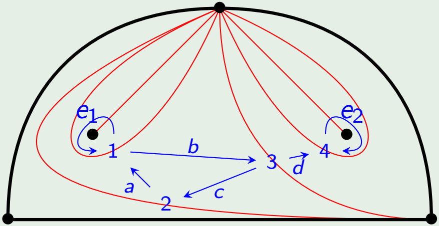
From triangulations to clannish algebras, I

- Fix an admissible triangulation Γ of $\mathbf{S}$. We associate a quiver $Q^\Gamma = (Q_0^\Gamma, Q_1^\Gamma)$ with Sp^Γ and Z^Γ as follows:
  - Q_0^Γ: the arcs in Γ whose endpoints are in the boundary.
  - $Q_1^\Gamma \setminus Sp^\Gamma$: the oriented angles between two edges of a triangle.
  - Sp^Γ: the loops in Γ enclosing a puncture.

From triangulations to clannish algebras, I

- Fix an admissible triangulation Γ of $\mathbf{S}$. We associate a quiver $Q^\Gamma = (Q_0^\Gamma, Q_1^\Gamma)$ with Sp^Γ and Z^Γ as follows:
  - Q_0^Γ: the arcs in Γ whose endpoints are in the boundary.
  - $Q_1^\Gamma \setminus Sp^\Gamma$: the oriented angles between two edges of a triangle.
  - Sp^Γ: the loops in Γ enclosing a puncture.
  - Z^Γ: $\mu\nu$ if μ and ν are induced from the angles in the same triangle.

Example



$$Sp = \{e_1, e_2\};$$

$$Z = \{ba, ac, cb\}.$$

From triangulations to clannish algebras, II

Proposition (Geiß–Labardini-Fragoso–Schröer; Qiu–Z)

For each admissible triangulation Γ *(which always exists), the algebra* $A^\Gamma = \mathbf{k}Q^\Gamma / Z^\Gamma \cup \{e^2 - e \mid e \in Sp^\Gamma\}$ *is a clannish algebra.*

From triangulations to clannish algebras, II

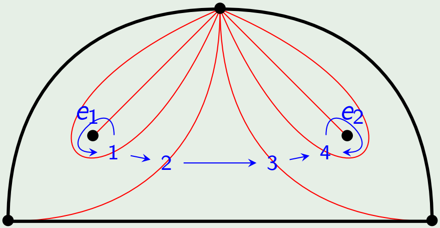
Proposition (Geiß–Labardini-Fragoso–Schröer; Qiu–Z)

For each admissible triangulation Γ *(which always exists), the algebra* $A^\Gamma = \mathbf{k}Q^\Gamma / Z^\Gamma \cup \{e^2 - e \mid e \in Sp^\Gamma\}$ *is a clannish algebra.*

Remark

The hereditary algebras of type A*,* D*,* $\widetilde{A}$ *and* $\widetilde{D}$ *can be obtained by this way.*

Example



$$Sp^\Gamma = \{e_1, e_2\};\ Z^\Gamma = \emptyset.$$

This clannish algebra is the hereditary algebra of type $\widetilde{D}_5$.

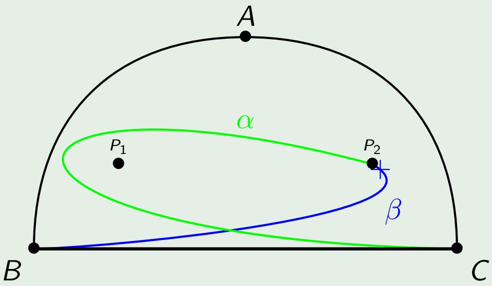
Tagged curves

- Tagged curve: a curve γ in $\mathbf{S}$ connecting marked points with an extra label (tagged or not) on each endpoint which is a puncture.

Tagged curves

- Tagged curve: a curve γ in **S** connecting marked points with an extra label (tagged or not) on each endpoint which is a puncture.

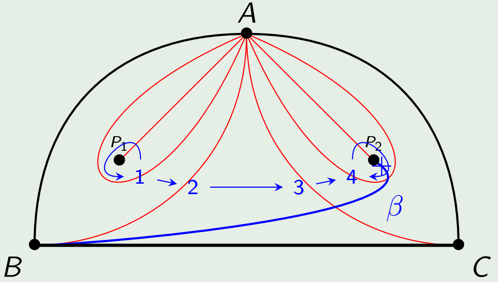
Example



From tagged curves to modules

Each tagged curve γ gives a walk $W(\gamma)$ (in the quiver Q^Γ) which corresponds to an indecomposable module $X(\gamma)$ of the clannish algebra. See the following example.

Example



$W(\beta)$: $3 \longrightarrow 4$

$X(\beta)$: $0 \circlearrowright 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbf{k} \xrightarrow{1} \mathbf{k} \circlearrowleft 1$

Correspondence

Theorem

The above map X from the set of tagged curves (up to homotopy) which are not in the triangulation Γ to the set of indecomposable modules (up to isomorphism) is injective.

Auslander-Reiten translation

- Recall that for each module M, its Auslander-Reiten translation τM is obtained by the following way:

Auslander-Reiten translation

- Recall that for each module M, its Auslander-Reiten translation τM is obtained by the following way:
  1. take its projective presentation: $P'' \xrightarrow{f} P' \to M \to 0$;

Auslander-Reiten translation

- Recall that for each module M, its Auslander-Reiten translation τM is obtained by the following way:
  1. take its projective presentation: $P'' \xrightarrow{f} P' \to M \to 0$;
  2. apply the Nakayama functor ν: $\nu P'' \xrightarrow{\nu f} \nu P'$;

Auslander-Reiten translation

- Recall that for each module M, its Auslander-Reiten translation τM is obtained by the following way:
  1. take its projective presentation: $P'' \xrightarrow{f} P' \to M \to 0$;
  2. apply the Nakayama functor ν: $\nu P'' \xrightarrow{\nu f} \nu P'$;
  3. take the kernel: $0 \to \tau M \to \nu P'' \xrightarrow{\nu f} \nu P'$.

Example

$$0 \circlearrowleft 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbf{k} \circlearrowright 0 \qquad \xrightarrow{f}$$

$$0 \circlearrowleft 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbf{k} \xrightarrow{\binom{1}{1}} \mathbf{k}^2 \circlearrowright \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \longrightarrow$$

$$0 \circlearrowleft 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbf{k} \xrightarrow{1} \mathbf{k} \circlearrowright 1 \qquad \longrightarrow 0$$

Example

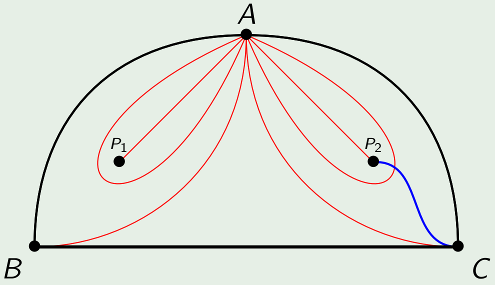
$$0 \longrightarrow \quad 0 \circlearrowright 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbf{k} \circlearrowleft 0$$

$$\longrightarrow \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circlearrowright \mathbf{k}^2 \xrightarrow{\binom{1}{1}} \mathbf{k} \xrightarrow{1} \mathbf{k} \xrightarrow{1} \mathbf{k} \circlearrowleft 0$$

$$\xrightarrow{\nu f} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circlearrowright \mathbf{k}^2 \xrightarrow{\binom{1}{1}} \mathbf{k} \xrightarrow{1} \mathbf{k} \xrightarrow{0} 0 \circlearrowleft 0$$

Example

$$\tau X(\beta):\ 0\ \circlearrowright\ 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbf{k}\ \circlearrowleft\ 0$$



tagged rotation

Definition

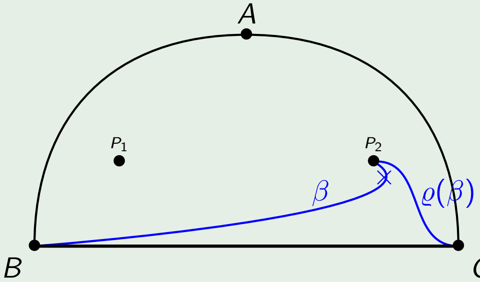
The tagged rotation ϱ on a tagged curve is changing the tagging at the end point which is a puncture and moving the end point in the boundary along the boundary anticlockwise to the next marked point.

tagged rotation

Definition

The tagged rotation ϱ on a tagged curve is changing the tagging at the end point which is a puncture and moving the end point in the boundary along the boundary anticlockwise to the next marked point.

Example



the tagged rotation becomes the Auslander-Reiten translation

Theorem

For each tagged curve $\gamma \notin \Gamma$, we have that

the tagged rotation becomes the Auslander-Reiten translation

Theorem

For each tagged curve $\gamma \notin \Gamma$, we have that

1. *if $X(\gamma)$ is not projective, then $\varrho(\gamma) \notin \Gamma$ and*

$$X(\varrho(\gamma)) \cong \tau X(\gamma);$$

the tagged rotation becomes the Auslander-Reiten translation

Theorem

For each tagged curve $\gamma \notin \Gamma$, we have that

1. *if $X(\gamma)$ is not projective, then $\varrho(\gamma) \notin \Gamma$ and*

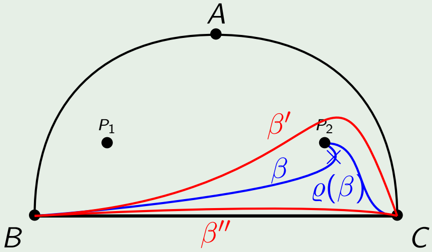
$$X(\varrho(\gamma)) \cong \tau X(\gamma);$$

2. *if $X(\gamma)$ is projective, then $\varrho(\gamma) \in \Gamma$, $\varrho^2(\gamma) \notin \Gamma$ and*

$$X(\varrho^2(\gamma)) \cong \nu X(\gamma).$$

Almost split sequences

Example



The almost split sequence ending at $X(\beta)$ is

$$0 \to X(\varrho(\beta)) \to X(\beta') \oplus X(\beta'') \to X(\beta) \to 0,$$

where $X(\beta'') = 0$ since it is homotopic to a boundary segment.

Intersection numbers

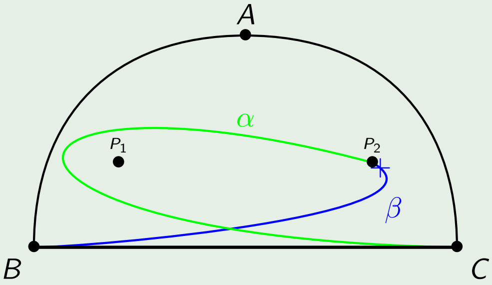
Definition

For any tagged curves γ_1, γ_2, the intersection number is defined as

$$\mathrm{Int}(\gamma_1, \gamma_2) := \mathrm{Int}^G(\gamma_1, \gamma_2) + \mathrm{Int}^T(\gamma_1, \gamma_2)$$

where $\mathrm{Int}^G(\gamma_1, \gamma_2)$ counts their intersections in their interiors and $\mathrm{Int}^T(\gamma_1, \gamma_2)$ counts their intersections at the endpoints where they have different tagged ways.

Example



$$\mathrm{Int}(\alpha, \beta) = 2$$

tagged arcs and tagged triangulations

Definition

tagged arcs and tagged triangulations

Definition

1. A tagged arc is a tagged curve γ with $\mathrm{Int}(\gamma, \gamma) = 0$.

tagged arcs and tagged triangulations

Definition

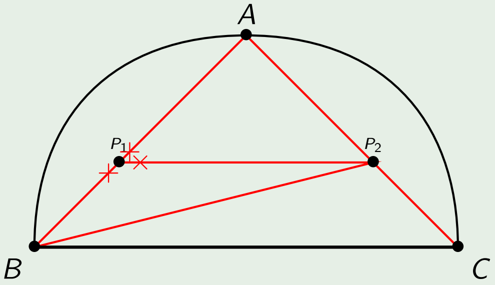
1. A tagged arc is a tagged curve γ with $\mathrm{Int}(\gamma, \gamma) = 0$.
2. A tagged triangulation of $\mathbf{S}$ is a maximal collection $\Gamma^\times$ of tagged arcs such that for any $\gamma_1, \gamma_2 \in \Gamma^\times$, $\mathrm{Int}(\gamma_1, \gamma_2) = 0$.

tagged arcs and tagged triangulations

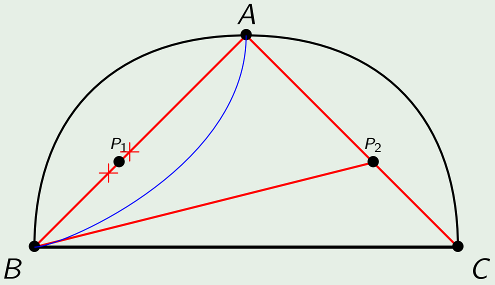
Definition

1. A tagged arc is a tagged curve γ with $\mathrm{Int}(\gamma, \gamma) = 0$.
2. A tagged triangulation of $\mathbf{S}$ is a maximal collection $\Gamma^\times$ of tagged arcs such that for any $\gamma_1, \gamma_2 \in \Gamma^\times$, $\mathrm{Int}(\gamma_1, \gamma_2) = 0$.
3. A flip of a tagged triangulation is replacing one tagged arc by another to get a new tagged triangulation.

Example



Example



τ-tilting theroy

Definition (Adachi-Iyama-Reiten, 2013)

Let A be a finite dimensional algebra and M be a finitely generated A-module.

τ-tilting theroy

Definition (Adachi-Iyama-Reiten, 2013)

Let A be a finite dimensional algebra and M be a finitely generated A-module.

1. We call M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

τ-tilting theroy

Definition (Adachi-Iyama-Reiten, 2013)

Let A be a finite dimensional algebra and M be a finitely generated A-module.

1. We call M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.
2. We call M τ-tilting if M is τ-rigid and $\mid M \mid = \mid A \mid$.

τ-tilting theroy

Definition (Adachi-Iyama-Reiten, 2013)

Let A be a finite dimensional algebra and M be a finitely generated A-module.

1. We call M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.
2. We call M τ-tilting if M is τ-rigid and $\mid M \mid = \mid A \mid$.
3. We call M support τ-tilting if there exists an idempotent e of A such that M is a τ-tilting $(A/\langle e \rangle)$-module.

τ-tilting theroy

Definition (Adachi-Iyama-Reiten, 2013)

Let A be a finite dimensional algebra and M be a finitely generated A-module.

1. We call M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.
2. We call M τ-tilting if M is τ-rigid and $| M |=| A |$.
3. We call M support τ-tilting if there exists an idempotent e of A such that M is a τ-tilting $(A/\langle e \rangle)$-module.
4. For each support τ-tilting module M, the mutation of M is defined.

Intersecion-dimension formula

Theorem

For any tagged curves $\gamma_1, \gamma_2 \notin \Gamma$, *we have*

$$Int(\gamma_1, \gamma_2) = \sum_{\{i,j\}=\{1,2\}} \dim Hom(X(\gamma_i), \tau X(\gamma_j)).$$

Bijections

Corollary

Bijections

Corollary

- *The map X above induces*

Bijections

Corollary

- *The map X above induces*
  1. *a bijection between the tagged arcs which are not in Γ and the indecomposable τ-rigid A-modules;*

Bijections

Corollary

- *The map X above induces*
  1. *a bijection between the tagged arcs which are not in Γ and the indecomposable τ-rigid A-modules;*
  2. *a bijection between the tagged triangulations of* $\mathbf{S}$ *which do not contain arcs in Γ and the τ-tilting A-modules; and*

Bijections

Corollary

- *The map X above induces*
  1. *a bijection between the tagged arcs which are not in Γ and the indecomposable τ-rigid A-modules;*
  2. *a bijection between the tagged triangulations of $\mathbf{S}$ which do not contain arcs in Γ and the τ-tilting A-modules; and*
  3. *a bijection between the tagged triangulations of $\mathbf{S}$ and the support τ-tilting A-modules.*

Bijections

Corollary

- *The map X above induces*
  1. *a bijection between the tagged arcs which are not in Γ and the indecomposable τ-rigid A-modules;*
  2. *a bijection between the tagged triangulations of $\mathbf{S}$ which do not contain arcs in Γ and the τ-tilting A-modules; and*
  3. *a bijection between the tagged triangulations of $\mathbf{S}$ and the support τ-tilting A-modules.*
- *Moreover, under the third bijection, the flip of tagged triangulations is compatible with the mutation of support τ-tilting modules.*

Thank you very much!

To cluster theory

- For each marked surface, there is a 2-Calabi-Yau triangulated category $\mathcal{C}$ satisfying the following properties:

To cluster theory

- For each marked surface, there is a 2-Calabi-Yau triangulated category $\mathcal{C}$ satisfying the following properties:
  1. for each admissible triangulation Γ, there exists a cluster tilting objects T in $\mathcal{C}$ such that

$$\mathcal{C}/\langle T \rangle \simeq \mathrm{mod} A^{\Gamma};$$

To cluster theory

- For each marked surface, there is a 2-Calabi-Yau triangulated category $\mathcal{C}$ satisfying the following properties:
  1. for each admissible triangulation Γ, there exists a cluster tilting objects T in $\mathcal{C}$ such that
  $$\mathcal{C}/\langle T\rangle \simeq \mathrm{mod} A^{\Gamma};$$
  2. the map X induces an injective map X' from the set of the tagged curves to the set of the indecomposable objects in $\mathcal{C}$ such that
  $$\mathrm{Int}(\gamma_1, \gamma_2) = \dim \mathrm{Ext}^1_{\mathcal{C}}(X'(\gamma_1), X'(\gamma_2)).$$