On symmetric invariants of semi-direct products

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The reductive case

Let $\mathfrak{g} = \mathrm{Lie}\, G$ be a complex reductive Lie algebra, $G = G^\circ$ connected simply connected, $\mathcal{S}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ the symmetric algebra of $\mathfrak{g}$. (Of course, $\mathfrak{g}^* \cong \mathfrak{g}$.)

Symmetric invariants — Chevalley restriction theorem

$\varphi\colon\ \mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{t}]^W$ is an isomorphism, W is a reflection group, and $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[H_1, \ldots, H_{\mathrm{rk}\mathfrak{g}}]$ is a polynomial ring in $\mathrm{rk}\mathfrak{g} = \dim \mathfrak{t}$ variables.

Example ($\mathfrak{gl}_n$ and elementary symmetric polynomials)

$\mathbb{C}[\mathfrak{gl}_n]^{\mathrm{GL}_n} = \mathbb{C}[\Delta_1, \ldots, \Delta_n]$ with $\det(A - \lambda E_n) = \sum(-1)^k \Delta_{n-k}(A)\lambda^k$ and $\varphi(\Delta_k) = \sigma_k$; $W = S_n$.

Question:

Are there other Lie algebras with a similar property?

Similar property: $\mathbb{C}[\mathfrak{q}^*]^Q$ is a polynomial ring in $\mathrm{ind}\mathfrak{q}$ variables.

Symmetric invariants and index

$\mathfrak{q} = \mathrm{Lie}\, Q$ is an algebraic Lie algebra, $Q = Q^\circ$, $\mathcal{S}(\mathfrak{q}) = \mathbb{C}[\mathfrak{q}^*]$, Q acts on $\mathfrak{q}$, on $\mathfrak{q}^*$, and on $\mathbb{C}[\mathfrak{q}^*]$, and so does $\mathfrak{q}$ (by derivations).

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} := \{H \mid \xi{\cdot}H = 0\ \forall \xi \in \mathfrak{q}\} = \mathcal{S}(\mathfrak{q})^Q = \mathbb{C}[\mathfrak{q}^*]^Q.$$

$\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} \cong Z\mathbf{U}(\mathfrak{q})$ (Duflo); $\mathfrak{q}^*$ is a Poisson variety, coadjoint orbits $Q\gamma \subset \mathfrak{q}^*$ are symplectic leaves, and each $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is constant on any $Q\gamma$.
In general, the coadjoint action is very important in representation theory.

Index of $\mathfrak{q}$ or of (ρ, V)

$\mathrm{ind}\mathfrak{q} := \min\limits_{\gamma\in\mathfrak{q}^*} \dim \mathfrak{q}_\gamma = \dim \mathfrak{q} - \max\limits_{\gamma\in\mathfrak{q}^*} \dim Q\gamma = \mathrm{tr.deg}\, \mathbb{C}(\mathfrak{q}^*)^Q$;
for a representation $\rho\colon\ \mathfrak{q} \to \mathfrak{gl}(V)$, $\mathrm{ind}(\mathfrak{q}, V) := \mathrm{tr.deg}\, \mathbb{C}(V^*)^{\mathfrak{q}}$.

For example, $\mathrm{ind}\mathfrak{g} = \mathrm{rk}\mathfrak{g}$; $\mathrm{ind}\mathfrak{q} = \dim \mathfrak{q}$ if and only if $\mathfrak{q}$ is Abelian.
Why do we want $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ to be a polynomial ring? Well, e.g. there is no need to describe the relations, or $\mathfrak{q}$ may have some other nice properties, like it happens with representations of finite or simple reductive groups.

Some answers

Examples (of $\mathfrak{q}$ such that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a polynomial ring in $\operatorname{ind}\mathfrak{q}$ variables)

1. commutative $\mathfrak{q}$, where $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{q})$;
2. a Frobenius Lie algebra (i.e., of index zero), here $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}$, the name was given by Ooms, no classification is known (possible);
3. a Heisenberg Lie algebra, $\mathfrak{q} = \mathbb{C}^n \oplus (\mathbb{C}^n)^* \oplus \mathbb{C}z$ with $[x_i, x_j^*] = \delta_{ij} z$, here $\operatorname{ind}\mathfrak{q} = 1$, $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}[z]$;
4. $\mathfrak{q}$ with $\operatorname{ind}\mathfrak{q} = 1$, providing $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} \neq \mathbb{C}$;
5. nilpotent $\mathfrak{q}$ with $\operatorname{ind}\mathfrak{q} = 2$ (follows from a result of Michel Brion [1983] on linear actions of unipotent groups);
6. $\mathfrak{q} = \mathfrak{b} \ltimes (\mathfrak{g}/\mathfrak{b})^{\mathrm{ab}}$, where $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra and $(\mathfrak{g}/\mathfrak{b})^{\mathrm{ab}}$ is an Abelian ideal of $\mathfrak{q}$ (Panyushev, Y);
7. the nilpotent radical of $\mathfrak{b}$ (Kostant), the truncated Borel $\mathfrak{b}_{\mathrm{tr}} \subset \mathfrak{b}$ (Joseph).

Other candidates

Some false conjectures

Premet's: $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is a polynomial ring in $\mathrm{ind}\,\mathfrak{g}_e = \mathrm{rk}\,\mathfrak{g}$ variables if $\mathfrak{g}_e \subset \mathfrak{g}$ is a centraliser of an element $e \in \mathfrak{g}$.

Joseph's: $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a polynomial ring in $\mathrm{ind}\,\mathfrak{q}$ variables if $\mathfrak{q}$ is a truncated bi-parabolic subalgebra of $\mathfrak{g}$.

Both conjectures are true in types A and C, but false in E_8 (Y 2007).

One conjecture, which I very much wanted to be true, but alas!

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a symmetric decomposition (i.e., a $\mathbb{Z}/2\mathbb{Z}$-grading), then one says that $(\mathfrak{g}, \mathfrak{g}_0)$ is a *symmetric pair*.
Let $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ be the corresponding *Z_2-contraction* of $\mathfrak{g}$.
Panyushev's conjecture : $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring in $\mathrm{ind}\,\tilde{\mathfrak{g}} = \mathrm{rk}\,\mathfrak{g}$ variables.

Contractions of Lie algebras

$\mathbb{Z}_2$-contraction: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \to \tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$ in $\tilde{\mathfrak{g}}$.

Let $\varphi_t \colon \mathfrak{g} \to \mathfrak{g}$ be a linear map with $\varphi_t|_{\mathfrak{g}_0} = \mathrm{id}$, $\varphi_t|_{\mathfrak{g}_1} = t\mathrm{id}$ ($t \in \mathbb{C}^\times$). Set

$$[\xi, \eta]_t := \varphi_t^{-1}([\varphi_t(\xi), \varphi_t(\eta)]) \text{ for } \xi, \eta \in \mathfrak{g}.$$

Then $[\ ,\ ]_t$ is a Lie bracket on $\mathfrak{g}$ and

$[\xi, \eta]_t = t[\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}_1$,

$[\xi, \eta]_t = [\xi, \eta]$ for $\xi \in \mathfrak{g}_0$, $\eta \in \mathfrak{g}$.

The Lie bracket of $\tilde{\mathfrak{g}}$ is $\lim_{t\to 0}[\ ,\ ]_t$.

Remark

Any vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus V$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, leads to a contraction $\mathfrak{g} \to \tilde{\mathfrak{g}} = \mathfrak{h} \ltimes (\mathfrak{g}/\mathfrak{h})^{\mathrm{ab}}$.

Contractions of Lie algebras and invariants

Let $\mathfrak{g} \to \tilde{\mathfrak{g}}$ be a contraction defined by φ_t.

Definition (highest t-component)

Extend φ_t to $\mathcal{S}(\mathfrak{g})$. Then $\varphi_t(H) = t^d H^\bullet + t^{d-1} H_{d-1} + \dots$, where $H^\bullet \neq 0$, for any non-zero $H \in \mathcal{S}(\mathfrak{g})$. We say that $H^\bullet$ is *the highest t-component* of H and set $\deg_t H := d$.

Lemma

If $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, then $H^\bullet \in \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$.

Proof.

Note that $H^\bullet = \lim_{t\to 0} t^d \varphi_t^{-1}(H)$. If $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, then $t^d \varphi_t^{-1}(H)$ (as well as $\varphi_t^{-1}(H)$) is a symmetric invariant of $(\mathfrak{g}, [\ ,\ ]_t)$ and its limit at 0 is a symmetric invariant of $\tilde{\mathfrak{g}}$. $\square$

Let $\mathfrak{g} \to \tilde{\mathfrak{g}}$ be a $\mathbb{Z}_2$-contraction. Then $\mathrm{ind}\,\tilde{\mathfrak{g}} = \mathrm{rk}\,\mathfrak{g}$.

Theorem (Panyushev 2007)

Suppose that $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[H_1, \ldots, H_{\mathrm{rk}\mathfrak{g}}]$, each H_i is homogeneous, and the $H_i^\bullet$'s are algebraically independent, then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}} = \mathbb{C}[H_1^\bullet, \ldots, H_{\mathrm{rk}\mathfrak{g}}^\bullet]$.

Example ($\mathfrak{g} = \mathfrak{so}_{2n+1} \to \mathfrak{so}_{2n} \ltimes \mathbb{C}^{2n}$)

Take $H_i = \Delta_{2i} \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with $1 \leqslant i \leqslant n$. Then $\forall i \deg_t H_i = 2$. Moreover, $H_i^\bullet|_{\mathfrak{g}_0^* \times \{x\}} = -\Delta_{2i-2}(\mathfrak{so}_{2n-1}) \in \mathcal{S}(\mathfrak{so}_{2n-1})$ if $x = (1, 0, \ldots, 0) \in \mathfrak{g}_1^*$.

$$\left(\begin{array}{c|cccc} & 1 & 0 & \cdots & 0 \\ \hline -1 & & & & \\ 0 & & & & \\ \vdots & & & \mathfrak{so}_{2n-1} & \\ 0 & & & & \end{array}\right)$$

$\sum \deg_t H_i = 2n = \dim \mathfrak{g}_1$.

Theorem (Y 2014)

The $H_i^\bullet$'s are algebraically independent if and only if $\sum \deg_t H_i = \dim \mathfrak{g}_1$.

Some positive results

Theorem (Panyushev 2007, Y 2014)

$\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring (in $\mathrm{rk}\,\mathfrak{g}$ variables) if (and only if) the restriction homomorphism $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1^*]^{\mathfrak{g}_0}$ is surjective.

Note that

(i) $\mathfrak{g}_1^* \cong \mathrm{Ann}(\mathfrak{g}_0) \subset \mathfrak{g}^*$ and $\varpi\colon\ \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[\mathfrak{g}_1^*]$ maps $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}}$ to $\mathbb{C}[\mathfrak{g}_1^*]^{\mathfrak{g}_0}$ (or one can define ϖ using the isomorphisms $\mathfrak{g}^* \cong \mathfrak{g}$, $\mathfrak{g}_1^* \cong \mathfrak{g}_1$);

(ii) $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0} = \mathbb{C}[\mathfrak{g}_1^*]^{\mathfrak{g}_0} \subset \mathbb{C}[\tilde{\mathfrak{g}}^*]^{\tilde{\mathfrak{g}}} = \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$, because $\mathfrak{g}_1 \lhd \tilde{\mathfrak{g}}$ is an Abelian ideal;

(iii) if $H \in \mathcal{S}(\mathfrak{g})$ is homogeneous and $H \neq 0$, then $H^\bullet \in \mathcal{S}(\mathfrak{g}_1)$ if and only if $H^\bullet = \varpi(H) \neq 0$.

If $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1^*]^{\mathfrak{g}_0}$ is surjective, then $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0} \subset \mathbb{C}[\{H_i^\bullet\}]$ for any generating set $\{H_i\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, and actually there is $\{H_i\}$ such that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}} = \mathbb{C}[\{H_i^\bullet\}]$.

Some negative results

Suppose that $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1^*]^{\mathfrak{g}_0}$ is not surjective. Then we say that $(\mathfrak{g}, \mathfrak{g}_0)$ is a "non-surjective" pair. Up to direct sums, there are four of them, found by Helgason:

$$(\mathcal{E}_6, \mathcal{F}_4), (\mathcal{E}_7, \mathcal{E}_6 \oplus \mathbb{C}), (\mathcal{E}_8, \mathcal{E}_7 \oplus \mathfrak{sl}_2), \text{ and } (\mathcal{E}_6, \mathfrak{so}_{10} \oplus \mathfrak{so}_2).$$

If $\mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{g}_1^*]^{\mathfrak{g}_0}$ is not surjective, then $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0} \not\subset \mathbb{C}[\{H^\bullet \mid H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}\}]$ and no equality $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}} = \mathbb{C}[H_1^\bullet, \ldots, H_{\mathrm{rk}\,\mathfrak{g}}^\bullet]$ is possible. However, this does not immediately imply that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is not a polynomial ring.

Example (a spherical contraction)

$\mathfrak{so}_8 \to \tilde{\mathfrak{g}} = \mathcal{G}_2 \ltimes V$ with $[V, V] = 0$ and $V = \mathbb{C}^7 \oplus \mathbb{C}^7$. Here $\mathrm{ind}\,\tilde{\mathfrak{g}} = 4$ and $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by four invariants of bi-degrees $(0, 2)$, $(0, 2)$, $(0, 2)$, $(2, 8)$ w.r.t. $\tilde{\mathfrak{g}} = \mathcal{G}_2 \oplus V$. In comparison, generators of $\mathcal{S}(\mathfrak{so}_8)^{\mathfrak{so}_8}$ have degrees 2, 4, 4, 6.

Let $\mathfrak{q}$ be a Lie algebra over a field $\mathbb{K}$ and $\rho\colon\ \mathfrak{q} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{q}$. Consider a new Lie algebra $\mathfrak{s} = \mathfrak{q} \ltimes V$, where

$$[\xi + v, \eta + u] = [\xi, \eta] + \rho(\xi)u - \rho(\eta)v \ \text{ for all } \xi, \eta \in \mathfrak{q}, v, u \in V.$$

Describing symmetric invariants of an arbitrary $\mathfrak{s}$ is a hopeless task, it is not even true that $\mathcal{S}(\mathfrak{s})^{\mathfrak{s}} \supset \mathbb{C}[V^*]^{\mathfrak{q}}$ is always finitely generated.
We begin with localised V-invariants. Set

$$\hat{\mathfrak{q}} := (\mathfrak{q} \otimes \mathbb{K}(V^*))^V = \left\{ \sum a_i \xi_i \mid \sum a_i \rho(\xi_i) v = 0 \ \forall v \in V \right\}.$$

If $x \in V^*$, then $\hat{\mathfrak{q}}(x) := \{\xi(x) \mid \xi \in \hat{\mathfrak{q}}, \xi(x) \text{ is defined}\} \subset \mathfrak{q}$.

Lemma (valid over any $\mathbb{K}$)

$\hat{\mathfrak{q}}$ is a Lie algebra over $\mathbb{K}(V^*)$ of dimension $\min\limits_{x \in V^*} \dim \mathfrak{q}_x$. Further, $\hat{\mathfrak{q}}(y) \subset \mathfrak{q}_y$ for any $y \in V^*$ and $\hat{\mathfrak{q}}(x) = \mathfrak{q}_x$, $\operatorname{ind}\mathfrak{q}_x = \operatorname{ind}\hat{\mathfrak{q}}$ for generic $x \in V^*$.

Further properties of $\hat{\mathfrak{q}}$ (in case $\mathbb{K} = \mathbb{C}$)

Assume $\mathfrak{q} = \mathrm{Lie}\, Q$, $Q = Q^{\circ}$ is a complex algebraic group acting on V.

$$\hat{\mathfrak{q}} = (\mathfrak{q} \otimes \mathbb{C}(V^*))^V = \{\sum a_i \xi_i \mid \sum a_i \rho(\xi_i) v = 0\ \forall v \in V\}$$

Lemma

$\mathcal{S}(\hat{\mathfrak{q}}) = \mathbb{C}[\mathfrak{s}^*]^V \otimes_{\mathbb{C}[V^*]} \mathbb{C}(V^*) = (\mathbb{C}[\mathfrak{s}^*] \otimes_{\mathbb{C}[V^*]} \mathbb{C}(V^*))^V$, $\mathrm{Quot}\, \mathcal{S}(\hat{\mathfrak{q}}) = \mathbb{C}(\mathfrak{s}^*)^V$.

Reason: for any $x \in V^*$, $(\mathfrak{q}^* \times \{x\}) / \exp(V) \cong \mathfrak{q}_x^*$ and $\mathbb{C}[\mathfrak{q}^* \times \{x\}]^V \cong \mathcal{S}(\mathfrak{q}_x)$.

Clearly, $\mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}} = \mathcal{S}(\hat{\mathfrak{q}})^Q \cap \mathbb{C}[\mathfrak{s}^*]$ and even better, $Z\mathbf{U}(\mathfrak{s}) = (Z\mathbf{U}(\hat{\mathfrak{q}}))^Q \cap \mathbf{U}(\mathfrak{s})$.
The difficulty: Q acts on $\hat{\mathfrak{q}}$ via non-$\mathbb{C}(V^*)$-linear Lie algebra automorphisms.

Let $\psi\colon \mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}} \to \mathbb{C}[\mathfrak{q}^* \times \{x\}]^{Q_x \ltimes V} \cong \mathcal{S}(\mathfrak{q}_x)^{Q_x}$ be the restriction homomorphism. It should be surjective under some mild assumptions on Q and V.

Stable actions

Suppose that Q is reductive and the action of Q on V^* is *stable*, i.e., generic Q-orbits $Qx \subset V^*$ are closed.

Generic orbits are closed — there is a non-empty open subset $U \subset V^*$ such that $Qx = \overline{Qx}$ for any $x \in U$, Q_x is reductive, and all Q_x are Q-conjugate.

One says that Q_x is a *generic stabiliser* of the Q-action on V^*.

Also elements of $\mathbb{C}[V^*]^Q$ separate generic orbits and therefore $\mathbb{C}(V^*)^Q = \mathrm{Quot}\,\mathbb{C}[V^*]^Q$.

Theorem (Y 2014)

$\psi\colon\ \mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}} \to \mathbb{C}[\mathfrak{q}^*\times\{x\}]^{Q_x\ltimes V}$ *is surjective and* $\mathbb{C}(\mathfrak{s}^*)^{\mathfrak{s}} = \mathrm{Quot}\,\mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}}$.

Corollary

$\mathrm{tr.deg}\,\mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}} = \mathrm{ind}\,\mathfrak{s} = \mathrm{ind}\,\hat{\mathfrak{q}} + \mathrm{ind}(\mathfrak{q}, V)$; $\mathbb{C}(\mathfrak{s}^*)^{\mathfrak{s}} = \mathrm{Quot}(\mathcal{S}(\hat{\mathfrak{q}})^Q)$.

Consequences of the theorem

Let $x \in V^*$ be generic — Q_x is a generic stabiliser, Qx is closed, in particular, Q_x is reductive, and $\psi\colon \mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}} \to \mathbb{C}[\mathfrak{q}^*\times\{x\}]^{Q_x\ltimes V}$ is surjective. Take $\{h_i\} \subset \mathbb{C}[\mathfrak{q}_x^*]^{Q_x} \cong \mathbb{C}[\mathfrak{q}^*\times\{x\}]^{Q_x\ltimes V}$ and choose some $H_i \in \mathcal{S}(\mathfrak{s})^{\mathfrak{s}}$ with $\psi(H_i) = h_i$.

If $\{h_i\}$ generate $\mathbb{C}(\mathfrak{q}_x^*)^{Q_x}$, then $\{H_i\}$ generate $\mathbb{C}(\mathfrak{s}^*)^{\mathfrak{s}}$ over $\mathbb{C}(V^*)^Q$.

If $\{h_i\}$ generate $\mathbb{C}[\mathfrak{q}_x^*]^{Q_x}$, then $\mathbb{C}(V^*)^Q[\{H_i\}] = \mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}}\otimes_{\mathbb{C}[V^*]^Q}\mathbb{C}(V^*)^Q$ and $\mathbb{C}[V^*]^Q[\{H_i\}] = \mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}}$ if and only if “there is no division" by $F \in \mathbb{C}[V^*]^Q$. Theoretically, this can be checked on Q-invariant divisors $D \subset V^*$ using degenerations $\mathfrak{q}_x \to \hat{\mathfrak{q}}(y)$ for generic $y \in D$. (Here $\dim\hat{\mathfrak{q}}(y) = \dim\mathfrak{q}_x$.)

Regular invariants and polynomial rings

Suppose that $\mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}}$ is a polynomial ring in $\mathrm{ind}\,\mathfrak{s}$ variables.
(I): Q is reductive, the Q-action on V^* is stable. Then

- $\mathbb{C}[V^*]^Q$ is a polynomial ring (in $\mathrm{ind}(\mathfrak{q}, V)$ variables),
- $\mathbb{C}[\mathfrak{q}_x^*]^{Q_x}$ is a polynomial ring (in $\mathrm{ind}\,\mathfrak{q}_x$ variables) for generic $x \in V^*$.

A function $H \in \mathbb{C}[\mathfrak{s}^*]$ is a *proper semi-invariant* of Q if it is an eigenvector of Q that is not a Q-invariant.
(II): Q has no proper semi-invariants in $\mathbb{C}[\mathfrak{s}^*]^V$. Then

- $\mathbb{C}[V^*]^Q$ is a polynomial ring in $\mathrm{ind}(\mathfrak{q}, V)$ variables,
- the restriction homomorphisms $\mathbb{C}[\mathfrak{s}^*]^{\mathfrak{s}} \to \mathbb{C}[\mathfrak{q}^* \times \{x\}]^{Q \ltimes V} \cong \mathcal{S}(\mathfrak{q}_x)^{Q_x}$ is surjective and $\mathcal{S}(\mathfrak{q}_x)^{Q_x} = \mathcal{S}(\mathfrak{q}_x)^{\mathfrak{q}_x}$ is a polynomial ring in $\mathrm{ind}\,\mathfrak{q}_x$ variables for generic $x \in V^*$.

Applications of general results to symmetric pairs $(\mathfrak{g}, \mathfrak{g}_0)$

Let $G_0 \subset G$ be the connected subgroup with $\mathrm{Lie}\, G_0 = \mathfrak{g}_0$. Then G_0 is reductive and the G_0-action on $\mathfrak{g}_1^* \cong \mathfrak{g}_1$ is stable.
Let L be a generic stabiliser of the G_0-action on $\mathfrak{g}_1^*$. Set $\ell = \mathrm{rk}\mathfrak{l}$, $r := \mathrm{ind}(\mathfrak{g}_0, \mathfrak{g}_1)$.

If $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring (in $\mathrm{ind}\tilde{\mathfrak{g}} = \mathrm{rk}\mathfrak{g} = \ell + r$ variables), then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}} = \mathbb{C}[H_1, \dots, H_\ell, F_1, \dots, F_r]$, where

- each H_i or F_j is bi-homogeneous w.r.t. $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,
- $F_1, \dots, F_r \in \mathcal{S}(\mathfrak{g}_1) = \mathbb{C}[\mathfrak{g}_1^*]$ (freely) generate $\mathbb{C}[\mathfrak{g}_1^*]^{G_0} \cong \mathbb{C}[\mathfrak{g}_1]^{G_0}$,
- the set $\{H_i|_{\mathfrak{g}_0^* \times \{x\}} \mid 1 \leqslant i \leqslant \ell\}$ generates $\mathcal{S}(\mathfrak{l})^{\mathfrak{l}} = \mathcal{S}(\mathfrak{l})^L$ for generic $x \in \mathfrak{g}_1^*$ with $(G_0)_x = L$,
- $\sum \deg H_i + \sum \deg F_j = (\dim \mathfrak{g} + \mathrm{rk}\mathfrak{g})/2$ (follows from a result of Joseph-Shafrir).

Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a *Cartan subspace* (i.e., maximal consisting of semisimple elements). Then $\dim \mathfrak{c} = r$, $\mathbb{C}[\mathfrak{g}_1]^{G_0} \xrightarrow{\sim} \mathbb{C}[\mathfrak{c}]^{W_{\mathfrak{c}}}$, where $W_{\mathfrak{c}} = N_{G_0}(\mathfrak{c})/L$ is a complex reflection group, and $L = G_{0,\mathfrak{c}} = \{g \in G_0 \mid gs = s \text{ for all } s \in \mathfrak{c}\}$.

Obtaining the degree inequality

In order to prove that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is not a polynomial ring, we may assume that it is, i.e., $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}} = \mathbb{C}[\{H_i\}, \{F_j\}]$ and then show that

$$\sum \deg H_i + \sum \deg F_j > (\dim \mathfrak{g} + \mathrm{rk}\,\mathfrak{g})/2.$$

The main ingredients in the proof are the following statements and objects

- the restrictions $H_i|_{\mathfrak{l}^* \times \{x\}}$ generate $\mathcal{S}(\mathfrak{l})^L$ for generic $x \in \mathfrak{c}^* \subset \mathfrak{g}_1^*$,
- $0 \neq H_i|_{\mathfrak{l}^* \oplus \mathfrak{c}^*} \in \mathbb{C}[\mathfrak{l}^* \oplus \mathfrak{c}^*]^{N_{G_0}(\mathfrak{c})} = (\mathcal{S}(\mathfrak{l})^L \otimes \mathcal{S}(\mathfrak{c}))^{W_\mathfrak{c}}$,
- “symmetric subpairs" and related contractions.

The above inequality holds for all four “non-surjective" pairs.

http://www.mccme.ru/~yakimova/z2contr-2014.pdf

Symmetric subpairs

Let $s \in \mathfrak{g}_1$ be semisimple. Then $\mathfrak{g}_s$ is reductive and $(\mathfrak{g}_s, \mathfrak{g}_{0,s})$ with $\mathfrak{g}_{0,s} = (\mathfrak{g}_0)_s$ is a symmetric pair. Let $\mathfrak{g}_s \to \tilde{\mathfrak{g}}_s$ be the corresponding contraction. Then $\tilde{\mathfrak{g}}_s$ is a subalgebra of $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_s \oplus \mathfrak{m}$, and $[\tilde{\mathfrak{g}}_s, \mathfrak{m}] \subset \mathfrak{m}$ in $\tilde{\mathfrak{g}}_s$. We identify $\tilde{\mathfrak{g}}_s^*$ with the annihilator of $\mathfrak{m}$ in $\tilde{\mathfrak{g}}^*$, which is a $\tilde{\mathfrak{g}}_s$-invariant subspace.

Let $\psi_s : \mathbb{C}[\tilde{\mathfrak{g}}^*]^{\tilde{\mathfrak{g}}} \to \mathbb{C}[\tilde{\mathfrak{g}}_s^*]$ be the restriction homomorphism. Then ψ_s is injective and its image lies in $\mathcal{S}(\tilde{\mathfrak{g}}_s)^{\tilde{\mathfrak{g}}_s}$.

If $(\mathfrak{g}, \mathfrak{g}_0)$ is one of the “non-surjective" pairs and $s \neq 0$, then $(\mathfrak{g}_s, \mathfrak{g}_{0,s})$ is one of the “surjective" pairs and the generators of $\mathcal{S}(\tilde{\mathfrak{g}}_s)^{\mathfrak{g}_s}$ are known explicitly. This provides a lot of information on the H_i's.

The degree inequality in case of $(\mathcal{E}_6, \mathcal{F}_4)$

Here $r = 2$, $W_{\mathfrak{c}}$ is of type A_2, $L = \mathrm{Spin}_8$, and there is $s \in \mathfrak{g}_1$ such that $(\mathfrak{g}_s, \mathfrak{g}_{0,s}) = (\mathfrak{so}_{10} \oplus \mathbb{C}s, \mathfrak{so}_9)$. In particular, $\deg F_1 + \deg F_2 = 5$,
$\sum \deg_{\mathfrak{g}_0} H_i = 16$.
Assume that $\deg_{\mathfrak{g}_0} H_1 = 2$, $\deg_{\mathfrak{g}_0} H_2 = \deg_{\mathfrak{g}_0} H_3 = 4$, $\deg_{\mathfrak{g}_0} H_4 = 6$.

Using the contraction $\mathfrak{g}_s \to \tilde{\mathfrak{g}}_s$, we obtain that $\deg_{\mathfrak{g}_1} H_1, H_4 \geqslant 6$, $\deg_{\mathfrak{g}_1} H_2, H_3 \geqslant 5$. The reason for the latter inequality is that the symmetric invariants Δ_4 and the Pfaffian P does not span a $W_{\mathfrak{c}}$-invariant subspace of $\mathcal{S}(\mathfrak{so}_8)^{\mathfrak{so}_8}$.

Summing up,

$$\sum \deg F_j + \sum \deg H_i \geqslant 5 + 16 + 6 + 5 + 5 + 6 = 43 > 42 = (6 + 78)/2.$$

And this proves that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is not a polynomial ring for the contraction of $\mathcal{E}_6$ related to $(\mathcal{E}_6, \mathcal{F}_4)$.