HOMOGENEOUS PROJECTIVE VARIETIES WITH TAME SECANT VARIETIES

(joint work with A. V. Petukhov)

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Definitions

$\mathbb{X} \subset \mathbb{P}(V)$ nondegenerate projective variety.

RANK: $R[v] = R_{\mathbb{X}}[v] = \min\{r \in \mathbb{N} : v = x_1 + \cdots + x_r,\ [x_j] \in \mathbb{X}\}$

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Remark: $R_{\mathbb{X}}$ and $\underline{R}_{\mathbb{X}}$ are $Aut(\mathbb{X})$-invariant, so $Aut(\mathbb{X})$ acts on $\mathbb{X}_r$ and Σ_r.

Classical examples: Matrices

$$\mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^n) \supset \mathit{Segre}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}) = \{[A] : \mathrm{rk} A = 1\} =: \mathbb{X}$$

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Further examples

Tensor rank:
$\mathbb{P}(V_1 \otimes ... \otimes V_k) \supset Segre(\mathbb{P}(V_1) \times ... \times \mathbb{P}(V_k)) =: \mathbb{X}$ simple tensors.

$$T = u_1 \otimes v_1 \otimes ... \otimes w_1 + ... + u_r \otimes v_r \otimes ... \otimes w_r$$

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Waring rank of polynomials:
$\mathbb{P}(S^k\mathbb{C}^n) \supset Ver_k(\mathbb{P}^{n-1}) =: \mathbb{X}$ powers of linear forms.

$$P = L_1^k + ... + L_r^k$$

$$Aut(\mathbb{X}) = SL_n$$

A wild one: the twisted cubic

$$V = S^3\mathbb{C}^2, \quad \mathbb{X} = Ver_3(\mathbb{P}^1) \subset \mathbb{P}(V), \quad G = Aut(\mathbb{X}) = SL_2.$$

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Orbits: $\mathbb{P}(V) = \mathbb{X} \sqcup \mathbb{X}_2 \sqcup \mathbb{X}_3$, where, if $x, y \in \mathbb{C}^2$ is a basis,

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Remark: $\overline{\mathbb{X}_3} = T\mathbb{X} \subset \Sigma_2 = \mathbb{P}(V)$
The tangental variety is the wilderness.

General setting

G semisimple complex Lie group

$V = V_\lambda$ irreducible G-module

$\mathbb{X} \subset \mathbb{P}(V)$ closed G-orbit; $\mathbb{X} = G[v_\lambda] \cong G/P$

Question: When is $\mathbb{X}$ tame?

Subminuscule varieties

$\mathbb{X} \subset \mathbb{P}(V)$ is called subminuscule if $Aut(\mathbb{X}) \to GL(V)$ is the semisimple part of the isotropy representation of an irreducible Hermitean symmetric space.

Theorem [Landsberg et al]

If $\mathbb{X} \subset \mathbb{P}(V)$ is subminuscule, then it is tame and

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Remark: If G acts spherically on $\mathbb{P}(V)$, then $\mathbb{X} \subset \mathbb{P}(V)$ is subminuscule and hence tame. Furthermore, $Max(R_{\mathbb{X}}) = \mathrm{rank}_{Aut(\mathbb{X})}(\mathbb{P}(V))$.

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3) If V_λ is tame and $\lambda = \pi_1 + \pi_2$, with π_j fundamental,
then G is transitive on $\mathbb{P}(V_{\pi_1})$ and on $\mathbb{P}(V_{\pi_2})$,
hence $G = SL_n$ (or Sp_n) and $V = \mathbb{C}^n$ or $S^2\mathbb{C}^n$ or $\mathfrak{sl}_n$,
or $G = SL_m \times SL_n$ (or $Sp_m \times Sp_n$)and $V = \mathbb{C}^m \otimes \mathbb{C}^n$.

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2) and 3) reduce the study to fundamental representations.

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5) Base cases:

$(SL_6, \Lambda^3\mathbb{C}^6)$,

$(SO_n, \Lambda^2\mathbb{C}^n)$, $(Spin_n, RSpin_n)$ for $n \leq 12$,

$(Sp_{2n}, \Lambda_0^2\mathbb{C}^{2n})$, $(Sp_6, \Lambda_0^3\mathbb{C}^6)$,

(F_4, V_{π_1}), (F_4, V_{π_2}),

(E_7, V_{π_1}).

Classification theorem (representations)

The tame irreducible representations are:

Group G	Representation V
SL_n	$\mathbb{C}^n, (\mathbb{C}^n)^*, (\Lambda^2\mathbb{C}^n), (\Lambda^2\mathbb{C}^n)^*,$ $S^2\mathbb{C}^n,\ (S^2\mathbb{C}^n)^*,\ \mathfrak{sl}_n$
SO_n	$\mathbb{C}^n, RSpin_n (n \leq 10)$
Sp_{2n}	$\mathbb{C}^{2n}, \Lambda^2_0\mathbb{C}^{2n},\ S^2\mathbb{C}^{2n} \cong \mathfrak{sp}_{2n}$
E_6	$\mathbb{C}^{27}, (\mathbb{C}^{27})^*$
F_4	$\mathbb{C}^{26}$
G_2	$\mathbb{C}^7$
$SL_m \times SL_n$	$\mathbb{C}^m \otimes \mathbb{C}^n$
$SL_m \times Sp_{2n}$	$\mathbb{C}^m \otimes \mathbb{C}^{2n}$
$Sp_{2m} \times Sp_{2n}$	$\mathbb{C}^{2m} \otimes \mathbb{C}^{2n}$

Classification theorem (varieties)

The tame homogeneous projective varieties are:

Notation for $\mathbb{X}$	Ambient $\mathbb{P}(V)$	$Aut(\mathbb{X})$	Max $R_{\mathbb{X}}$
$\mathbb{P}(\mathbb{C}^n)$	$\mathbb{P}(\mathbb{C}^n)$	SL_n	1
$\mathrm{Ver}_2(\mathbb{P}(\mathbb{C}^n))$	$\mathbb{P}(\mathrm{S}^2\mathbb{C}^n)$	SL_n	n
$\mathrm{Gr}_2(\mathbb{C}^n)$	$\mathbb{P}(\Lambda^2\mathbb{C}^n)$	SL_n	$\lfloor \frac{n}{2} \rfloor$
$\mathrm{Fl}(1, n-1; \mathbb{C}^n)$	$\mathbb{P}(\mathfrak{sl}_n)$	SL_n	n
Q^{n-2}	$\mathbb{P}(\mathbb{C}^n)$	SO_n	2
S^{10}	$\mathbb{P}(\mathbb{C}^{16})$	$Spin_{10}$	2
$\mathrm{Gr}_\omega(2, \mathbb{C}^{2n})$	$\mathbb{P}(\Lambda_0^2\mathbb{C}^{2n})$	Sp_{2n}	n
E^{16}	$\mathbb{P}(\mathbb{C}^{27})$	E_6	3
F^{15}	$\mathbb{P}(\mathbb{C}^{26})$	F_4	3
$\mathrm{Segre}(\mathbb{P}(\mathbb{C}^m) \times \mathbb{P}(\mathbb{C}^n))$	$\mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$	$SL_m \times SL_n$	$\min\{m, n\}$

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3) If $\mathbb{X} \subset \mathbb{P}(V)$ is tame, then $I(\Sigma_r)$ is generated in degree $r+1$ by the $(r-1)$-th prolongation $(I_2(\mathbb{X}) \otimes S^{r-1}V^*) \cap S^{r+1}V^*$.

THE END

THANK YOU FOR THE ATTENTION!