What is modular category $\mathcal{O}$?

Wolfgang Soergel

Mathematisches Insitut
Universität Freiburg

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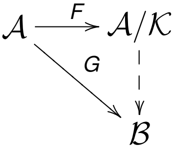
Given an abelian category $\mathcal{A}$ and a set of objects $\mathcal{K} \subset \mathcal{A}$ there always exists an exact functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{K}$ to an abelian category $\mathcal{A}/\mathcal{K}$ such that F annihilates all objects of $\mathcal{K}$ and that every exact functor $G : \mathcal{A} \to \mathcal{B}$ of abelian categories annihilating all objects of $\mathcal{K}$ factorizes uniquely over F. We call $\mathcal{A}/\mathcal{K}$ the **quotient category**.



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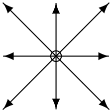
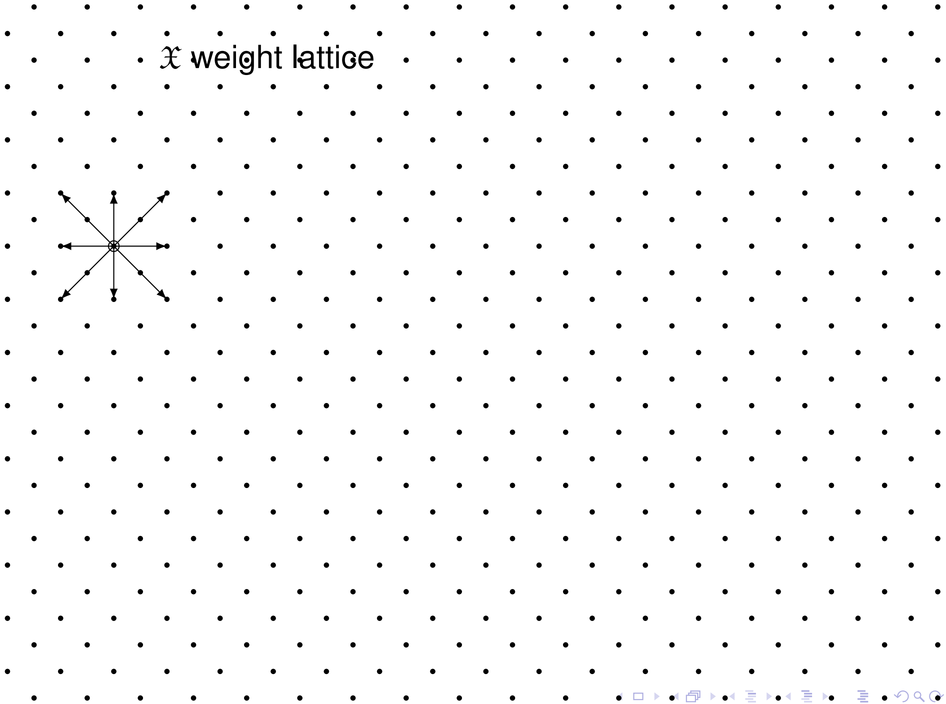
Example $G = \mathrm{Sp}(4; k)$



$\mathfrak{X}$ weight lattice



$\mathfrak{X}^+$ dominant weights

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$$\lambda \mapsto L(\lambda) \text{ the simple representation } L \text{ with highest weight } \lambda \text{ alias } \mathrm{Hom}_B(L, k_\lambda) \neq 0$$

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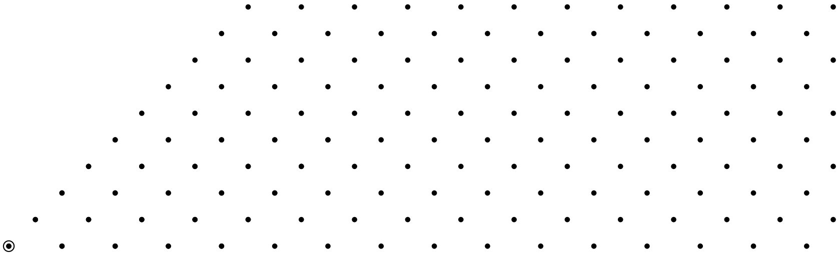
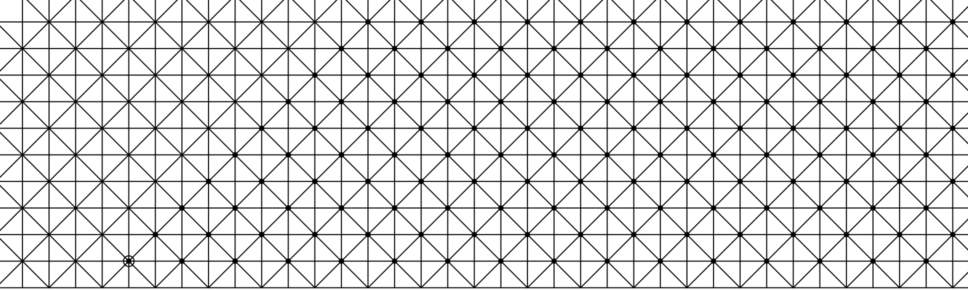


Consider affine Weyl group $\mathcal{W} := W \ltimes \langle R \rangle$



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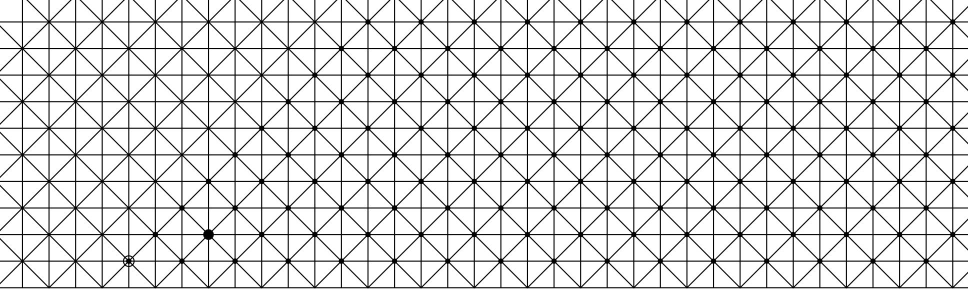

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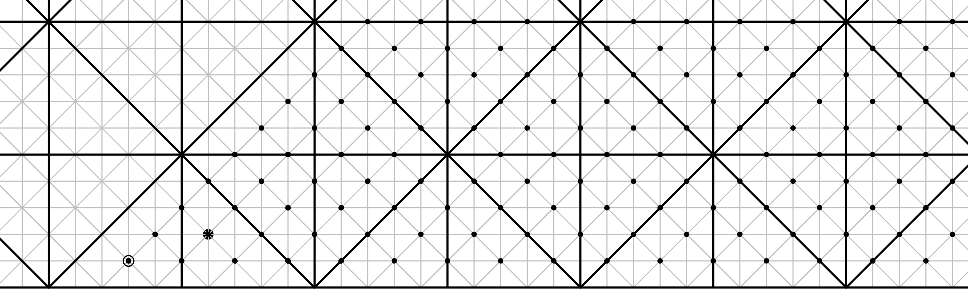
New $\mathcal{W}$-action $x \cdot_p \lambda := pxp^{-1}(\lambda + \rho) - \rho$, here for $p = 5$



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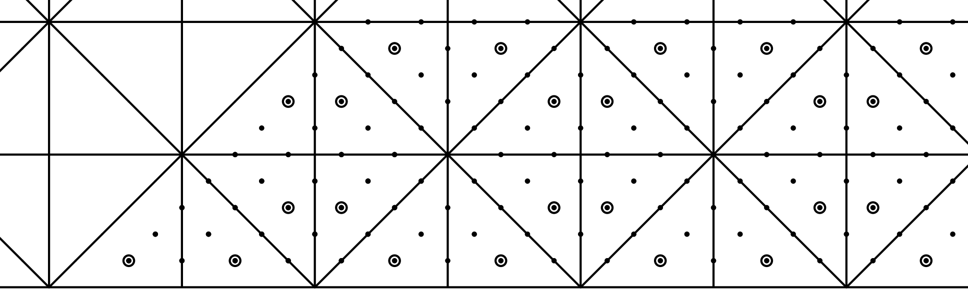
Linkage principle: The representation category decomposes

$$G\text{-Modf} = \bigoplus_{\Omega \in (\mathcal{X}/(\mathcal{W}\cdot_p))} G\text{-Modf}_\Omega$$



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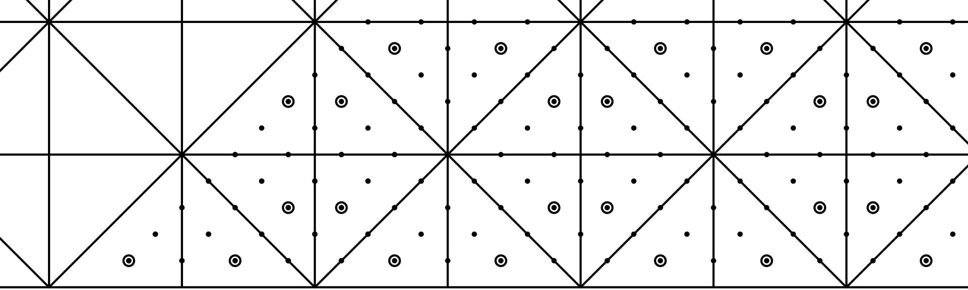

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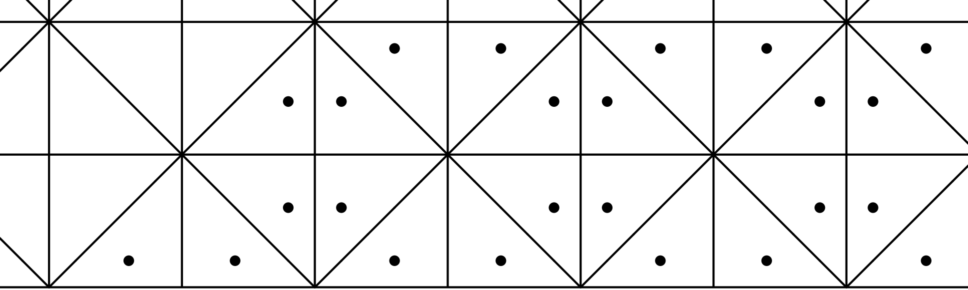

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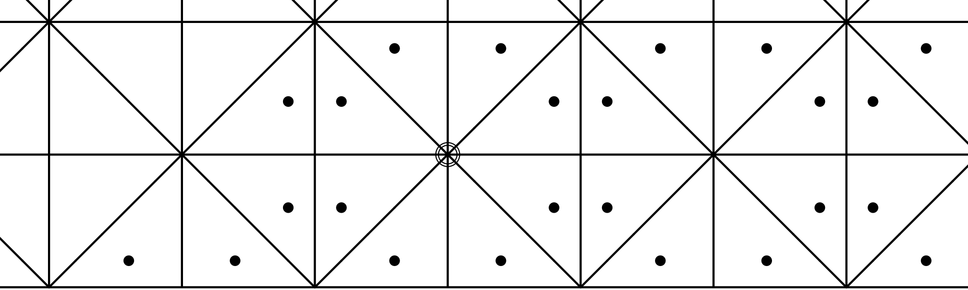
◎ Take st $:= (p-1)\rho$ the Steinberg point



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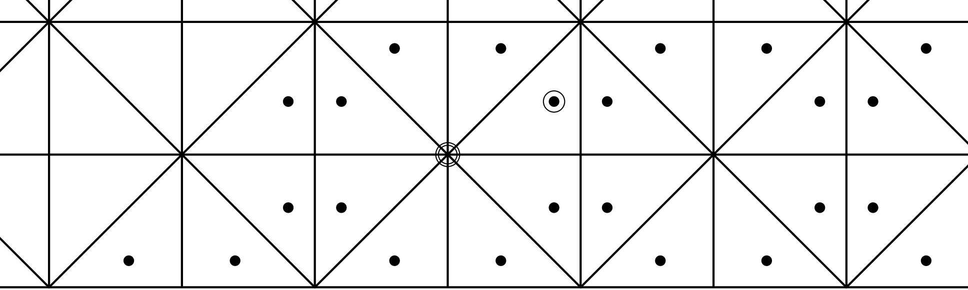

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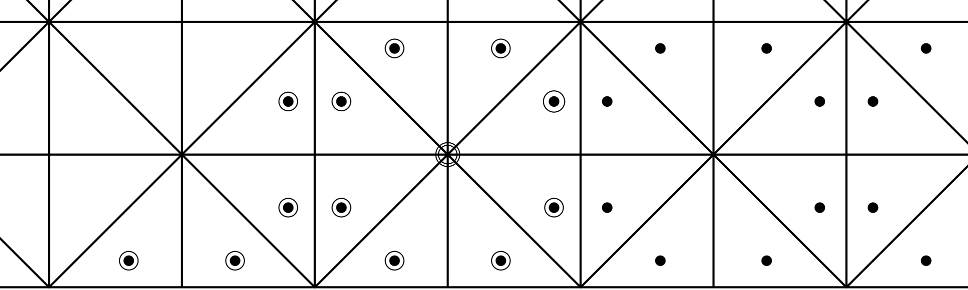

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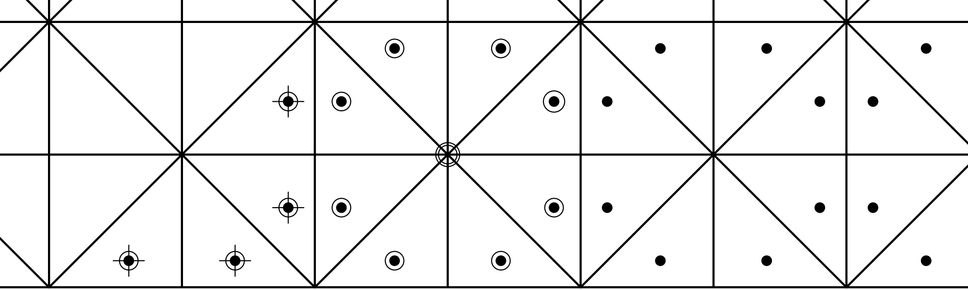

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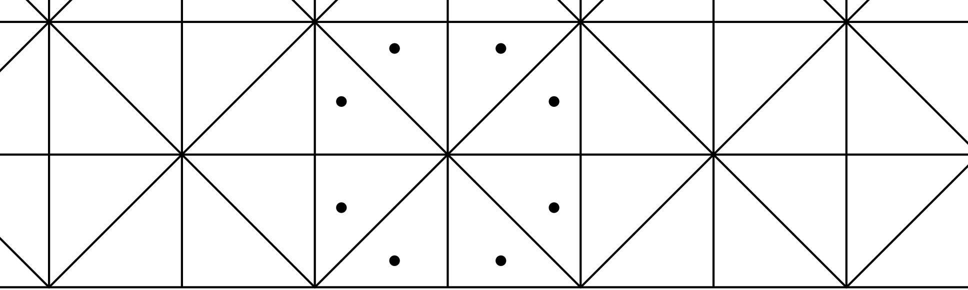

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(1) Simple objects are parametrized by the finite Weyl group;
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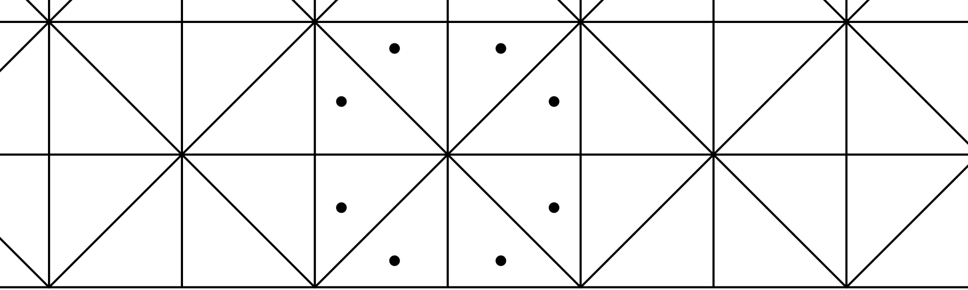
Modular $\mathcal{O}$ is relevant, since $[\nabla(\nu) : \bar{L}(\mu)] = [\mathrm{ind}_B^G k_\nu : L(\mu)]$



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- It admits an exact functor $\mathbb{V} : \mathcal{O} \to C$-Mod fully faithful on injectives.
- Here C is the nilfibre algebra (formerly called coinvariant algebra) with coefficients in k, so

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Significance of modular category $\mathcal{O}$:

- Lusztig's conjecture would imply that length of the Krull-Schmidt decomposition of these tensor products is independent of the characteristic of k.
- Consider $G = \mathrm{GL}(n; k)$ with C the quotient of the polynomial ring $k[X_1, \ldots, X_n]$ by the ideal generated by the symmetric polynomials of positive degree. Williamson showed that for $n = 4m + 7$ the number of summands will not yet be stable for any prime factor p of the Fibonacci numbers F_m, F_{m+1}.

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