Combinatorial aspects of exceptional sequences on (rational) surfaces

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Markus Perling

Universität Bielefeld
perling@math.uni-bielefeld.de

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Setup

We are interested in $D^b(X)$, the derived category of coherent sheaves over a smooth complete algebraic variety X over an algebraically closed field $\mathbb{K}$.

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We consider the case where X is a (rational) surface.

$\mathrm{CH}^*(X)$, $K_0(X)$ finitely generated, torsion free,
$q = p_g = 0$.

Exceptional sequences

Definition

- An object $\mathcal{E}$ in $D^b(X)$ is called *exceptional* if $\mathrm{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{E}[k]) = 0$ for $k \neq 0$ and $\mathrm{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{E}) = \mathbb{K}$.
- A sequence of objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is called *exceptional sequence* if every $\mathcal{E}_i$ is exceptional and $\mathrm{Hom}_{D^b(X)}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0$ for all $i > j$ and all k.
- A collection of objects in $D^b(X)$ is called *full* if it generates $D^b(X)$.
- An exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is called *strongly exceptional* if $\mathrm{Hom}_{D^b(X)}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0$ for all i, j and all $k \neq 0$.

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If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is a full strongly exceptional sequence, then the direct sum $\mathcal{T} = \bigoplus_{i=1}^n \mathcal{E}_i$ is a *tilting object*, i.e. for $A := \operatorname{End}_{D^b(X)}(\mathcal{T})$, the functor

$$\operatorname{RHom}(\mathcal{T}, \_) : \mathcal{T} \longrightarrow D^b(A - \operatorname{mod})$$

is an equivalence of categories.

A full exceptional sequence sequence is a special case of a semi-orthogonal decomposition of $D^b(X)$:

$$D^b(X) = \langle D_1, \dots, D_n \rangle, \text{ where } D_i = \langle \mathcal{E}_i \rangle \cong D^b(\mathbb{K} - \textbf{vect})$$
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It can (vaguely) be considered as a categorical analogue to cell decomposition.

In general, semi-orthogonal decompositions seem to be deeply connected to the geometry of X, but they are not very well understood.

In recent years, many interesting examples of such decompositions have been constructed (e.g. cubic threefolds, phantoms, ...).

Open problems I want to address:

- The classification of exceptional objects and sequences.
- The rationality conjecture: does the existence of a full exceptional sequence imply that X is rational?
- Relations to representation theory via
  - tilting,
  - singularity theory.

Example

A classical result is the classification of exceptional vector bundles on $\mathbb{P}^2$ by Drezet and Le Potier '85.

It turns out that every exceptional object on on $\mathbb{P}^2$ is locally free (up to shift in $D^b(\mathbb{P}^2)$) and can be included in a strongly exceptional sequence.

Rudakov '88

Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be an exceptional sequence on $\mathbb{P}^2$ and denote $e_i := \operatorname{rk} \mathcal{E}_i$. Then the e_i satisfy the **Markov** equation:

$$e_1^2 + e_2^2 + e_3^2 = 3 e_1 e_2 e_3.$$

The solutions to the Markov equations correspond essentially to exceptional sequences on $\mathbb{P}^2$.

Example

The solutions of the Markov equation can be exhausted inductively as follows:

The fundamental solution is

$$(e_1, e_2, e_3) = (1, 1, 1).$$

If (e_1, e_2, e_3) is a solution, then so are:

$$(e_2, e_1', e_3), (e_1, e_3', e_2),$$

where

$$e_1' = 3e_3e_2 - e_1 \qquad \text{right mutation of } e_1$$
$$e_3' = 3e_1e_2 - e_3 \qquad \text{left mutation of } e_3.$$

Conversely, any solution of the Markov equation can be reduced to the fundamental solution by mutation.

Mutations

Let $\mathcal{E}, \mathcal{F}$ be an exceptional pair in $D^b(X)$. Then we have two canonical triangles

$$L_{\mathcal{E}}\mathcal{F} \longrightarrow R\mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{K}} \mathcal{E} \xrightarrow{ev} \mathcal{F}$$

$$\mathcal{E} \xrightarrow{ev^*} R\mathcal{H}om(\mathcal{E}, \mathcal{F})^* \otimes_{\mathbb{K}} \mathcal{F} \longrightarrow R_{\mathcal{F}}\mathcal{E}$$

Then $L_{\mathcal{E}}\mathcal{F}, \mathcal{E}$ and $\mathcal{F}, R_{\mathcal{F}}\mathcal{E}$ are exceptional pairs again.

Definition

$L_{\mathcal{E}}\mathcal{F}$ and $R_{\mathcal{F}}\mathcal{E}$ are called *left-* and *right-mutation* of the pair $\mathcal{E}, \mathcal{F}$.

More generally, if $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is an exceptional sequence, then so are the mutations

$$\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, L_{\mathcal{E}_i}\mathcal{E}_{i+1}, \mathcal{E}_i, \mathcal{E}_{i+1}, \ldots, \mathcal{E}_n$$
$$\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, \mathcal{E}_{i+1}, R_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+1}, \ldots, \mathcal{E}_n$$

Example (continued)

For any $k \in \mathbb{Z}$, the following is a full exceptional sequence on $\mathbb{P}^2$:

$$\mathcal{O}(k-1), \mathcal{O}(k), \mathcal{O}(k+1).$$

We can create new exceptional sequences via mutation. In particular, if $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ is an exceptional sequence, then

$$\operatorname{rk} L_{\mathcal{E}_2}\mathcal{E}_3 = 3e_1e_2 - e_3 \text{ and } \operatorname{rk} R_{\mathcal{E}_2}\mathcal{E}_1 = 3e_3e_2 - e_1$$

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In particular, Rudakov's theorem entails:

- Every exceptional sequence on $\mathbb{P}^2$ can by mutation be transformed to an exceptional sequence of invertible sheaves of the above form.
- Up to $k \in \mathbb{Z}$, the exceptional sequence on $\mathbb{P}^2$ correspond precisely to the solutions of the Markov equation.

Example (continued)

All exceptional sequences on $\mathbb{P}^2$ are strongly exceptional and therefore give rise to a tilting object. For given $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, the corresponding endomorphism algebra can be described by a quiver

$$\bullet \xrightarrow{3e_3} \bullet \xrightarrow{3e_1} \bullet,$$

with relations given by $\mathrm{Hom}(\mathcal{E}_1, L_{\mathcal{E}_2}\mathcal{E}_3)$, where we have the following short exact sequence:

$$0 \to \mathrm{Hom}(\mathcal{E}_1, L_{\mathcal{E}_2}\mathcal{E}_3) \to \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \otimes_{\mathbb{K}} \mathrm{Hom}(\mathcal{E}_2, \mathcal{E}_3) \to \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_3) \to 0.$$

More on $\mathbb{P}^2$

Hacking and Prokhorov '10 consider $\mathbb{Q}$-Gorenstein degenerations of $\mathbb{P}^2$:

$$\mathcal{X} \longrightarrow \mathbb{C},$$

i.e. flat 1-parameter families where the total space $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein and whose general fiber is isomorphic to $\mathbb{P}^2$.

By definition, the special fiber $Y = \mathcal{X}_0$ has at most T-singularities (see below).

They show that for $K_Y^2 = 9$, the special fibers are of the form

$$Y \cong \mathbb{P}(e_1^2, e_2^2, e_3^2), \text{ where } e_1^2 + e_2^2 + e_3^2 = 3e_1 e_2 e_3.$$

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Theorem (Hacking '13)

There is a bijection between the set of exceptional vector bundles on $\mathbb{P}^2$ (modulo twist and dualizing) and the isomorphism classes of normal surfaces with isolated quotient singularity which admit a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$.

The main results

Let X be a (rational) surface.

Theorem

Let $\mathbf{E} = \mathcal{E}_1, \ldots, \mathcal{E}_n$ *be an exceptional sequence whose length equals* $\operatorname{rk} K_0(X)$ *such that* $\operatorname{rk} \mathcal{E}_i =: e_i \neq 0$ *for every* i*. Then to this sequence there is in a natural way associated a complete toric surface* $Y(\mathbf{E})$ *with* n *torus fixpoints which are either smooth (if* $e_i^2 = 1$*) or T-singularities of type* $\frac{1}{e_i^2}(1, k_i e_i - 1)$*, where* $\gcd\{k_i, e_i\} = 1$ *(if* $e_i^2 > 1$*). Moreover, this correspondence induces a natural isomorphism of Chow rings* $\mathrm{CH}^*(Y(\mathbf{E}))_{\mathbb{Q}} \to \mathrm{CH}^*(X)_{\mathbb{Q}}$*.*

Theorem

Any exceptional sequence on X *can be transformed by mutation into an exceptional sequence consisting only of objects of rank one.*

T-singularities

T-singularities were introduced '81 by Wahl. They can be defined by the property that they admit a 1-parameter $\mathbb{Q}$-Gorenstein smoothing.

Theorem (Wahl '81, Kollar – Shepherd-Barron '88)

A T-singularity is either a rational double point or a cyclic quotient singularity of type

$$\frac{1}{de^2}(1, kde - 1),$$

where $\gcd(k, e) = 1$.

The construction of $Y(\mathbf{E})$, I

The construction is purely combinatorial and a pretty shameless exploit of the Riemann-Roch formula.

By definition, for any $\mathcal{E}, \mathcal{F}$ in $D^b(X)$, the Euler form is given by

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \operatorname{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}[k]).$$

If $\mathcal{E}, \mathcal{F}$ are exceptional, then the Riemann-Roch formula states:

$$\chi(\mathcal{E}, \mathcal{F}) = -\frac{1}{2} K_X c_1(\mathcal{E}, \mathcal{F}) + \frac{1}{2ef}(c_1(\mathcal{E}, \mathcal{F})^2 + e^2 + f^2)$$

where e, f are the ranks and $c_1(\mathcal{E}, \mathcal{F}) = ec_1(\mathcal{F}) - fc_1(\mathcal{E})$.

$$\chi(\mathcal{E}, \mathcal{F}) + \chi(\mathcal{F}, \mathcal{E}) = \frac{1}{ef}(c_1(\mathcal{E}, \mathcal{F})^2 + e^2 + f^2)$$
$$\chi(\mathcal{E}, \mathcal{F}) - \chi(\mathcal{F}, \mathcal{E}) = -K_X c_1(\mathcal{E}, \mathcal{F}).$$

The construction of $Y(\mathbf{E})$, II

For any $1 \leq i < n$ we set

$$A_i := \frac{1}{e_i e_{i+1}} c_1(\mathcal{E}_i, \mathcal{E}_{i+1})$$

and

$$A_n := -K_X - \sum_{i=1}^{n-1} A_i$$

such that $A_i \in \mathrm{CH}^1(X)_{\mathbb{Q}} = \mathrm{CH}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $1 \leq i \leq n$.

By the Riemann-Roch formula we get:

1. $A_{i-1} \cdot A_i = \frac{1}{e_i^2}$ for all i.
2. $A_i \cdot A_j = 0$ else.
3. $\sum_{i=1}^{n} A_i = -K_X$.

This generalizes the toric systems of [Hille - P. '11] for the case of line bundles.

The construction of $Y(\mathbf{E})$, III

The final ingredient is the *Gale transform*. If we denote $A = \langle A_1, \ldots, A_n \rangle \subset \mathrm{CH}^1(X)_{\mathbb{Q}}$, we have a short exact sequence:

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{L} \mathbb{Z}^n \xrightarrow{c} A \longrightarrow 0,$$

where c maps the i-th element of the standard basis of $\mathbb{Z}^n$ to A_i and L can be represented as a row matrix with rows $l_1, \ldots, l_n \in \mathbb{Z}^2$.

Claim:

The l_i encode the combinatorial data of a toric surface as stated in the theorem.

Toric surfaces

A toric surface Y is a normal equivariant completion of a 2-dimensional algebraic torus $T \cong (\mathbb{K}^*)^2$, i.e. T acts on Y and embeds into Y as an open dense orbit.

We have

$$Y \setminus T = \cup_{i=1}^{n} D_i, \quad \text{where} \quad D_i \cong \mathbb{P}^1 \quad \forall i$$

The D_i form a cycle, i.e. $D_i \cdot D_{i+1} \neq 0$ for all i and $D_i \cdot D_j = 0$ otherwise.

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A toric surface can be described purely by combinatorial data which is given by primitive lattice vector $l_1, \ldots, l_n \in \mathbb{Z}^2$. These lattice vectors are cyclically arranged in $\mathbb{Z}^2$ and for each i, we have a relation

$$\alpha_i l_{i-1} + \beta_i l_i + \alpha_{i+1} l_{i+1} = 0,$$

where $\alpha_i, \beta_i \in \mathbb{Q}$ with

- $\alpha_i = D_{i-1} \cdot D_i = 1/\det(l_{i-1}, l_i)$,
- $\beta_i = D_i^2 = \alpha_{i-1}\alpha_i \det(l_{i+1}, l_{i-1})$

for every i.

For the l_i coming from our previous construction, we can now compare locally their properties with what we'd like to have for toric surfaces.

E.g. it's easy to see that $\det(l_{i-1}, l_i) = e_i^2$ for all i.

However, the following necessary conditions are not yet obvious:

- the primitivity of the l_i,
- their overall configuration,
- the precise types of singularities of $Y(\mathbf{E})$.

The idea is to use mutations.

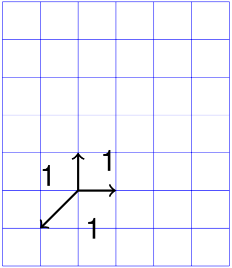
Mutations, I

Lemma

A mutation of $\mathcal{E}_1, \ldots, \mathcal{E}_n$ affects only one of the l_i.

Example

$\mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)$



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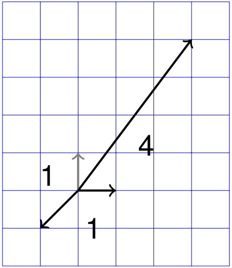

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$Y(\mathbf{E}) \cong \mathbb{P}(4, 1, 1)$

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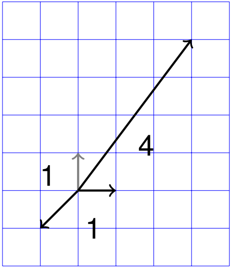

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The toric system for $\mathcal{O}(1), \mathcal{T}, \mathcal{O}(2)$ is given by $A_1, A_2, A_3 = \frac{1}{2}H, \frac{1}{2}H, 2H$, where H denotes the class of a hyperplane in $\mathbb{P}^2$.

Mutations, II

For any i, set $w_i := \frac{1}{e_i}(l_i - l_{i-1})$. We can show:

- The w_i are integral.
- $\det(w_i, w_{i+1}) = \chi(\mathcal{E}_i, \mathcal{E}_{i+1})$.

Lemma

Assume that $\det(l_{i-1}, l_{i+1}) > 0$ *and* $\det(w_i, w_{i+1}) > 0$. *Then either* $(\operatorname{rk} L_{\mathcal{E}_i}\mathcal{E}_{i+1}) < \max\{e_i^2, e_{i+1}^2\}$ *or* $(\operatorname{rk} R_{\mathcal{E}_{i+1}}\mathcal{E}_i) < \max\{e_i^2, e_{i+1}^2\}$.

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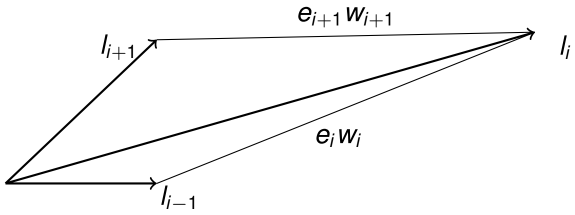

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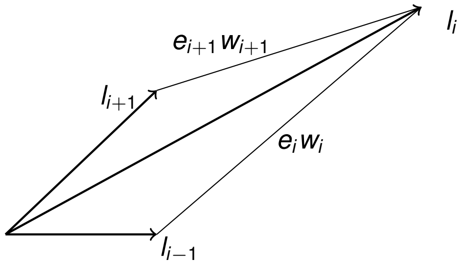

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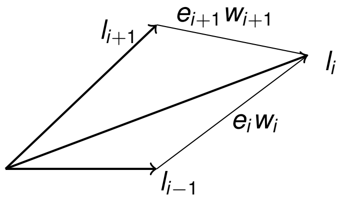

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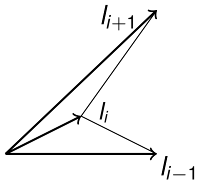

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Using this procedure, we can successively:

- Minimize the ranks of the objects in the sequence.
- Play inner and outer convexity of the l_i against each other.

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- Play inner and outer convexity of the l_i against each other.

By this and the Hodge index theorem, we finally get:

Theorem

Any exceptional sequence on X can be transformed by mutation into an exceptional sequence of one of the following shapes:

1. $\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-4}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$, *where* $\operatorname{rk} \mathcal{Z}_i = 0$ *for all i and* $\operatorname{rk} \mathcal{L}_i = 1$ *for all i.*
2. $\mathcal{Z}_1, \ldots, \mathcal{Z}_{n-3}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, *where* $\operatorname{rk} \mathcal{Z}_i = 0$ *for all i and the ranks of the $\mathcal{E}_i$ satisfy the Markov equation.*

Objects of rank 0

Example

Let $b : X \to \mathbb{P}^2$ be a blowup in one point with exceptional divisor E. Then the following is a full exceptional sequence on X:

$$\mathcal{O}_E(E), b^*\mathcal{T}, b^*\mathcal{O}(2), b^*\mathcal{O}(4).$$

Mutation yields a sequence

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where $\mathcal{R}$ has rank -4.

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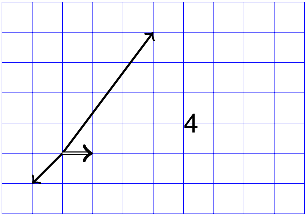

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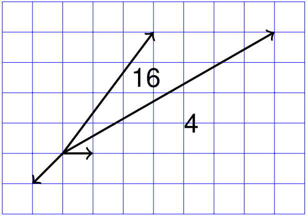
where $\mathcal{R}$ has rank -4.

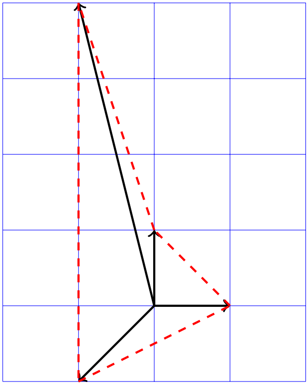
$\mathcal{O}_E(E), b^*\mathcal{T}, b^*\mathcal{O}(2), b^*\mathcal{O}(4)$

$b^*\mathcal{T}, \mathcal{R}, b^*\mathcal{O}(2), b^*\mathcal{O}(4)$



The shape of the $Y(\mathbf{E})$

1. The lattice volumes are $\det(l_{i-1}, l_i) = e_i^2$,
2. the line segments $l_i - l_{i-1}$ have lattice length $|e_i|$,
3. $K_Y^2 = K_X^2 = 12 - n$.



What have we achieved?

- The classification of exceptional objects is reduced to the classification of such objects of rank one. Unfortunately, exceptional objects of rank one are rarely invertible.
- There is now a combinatorial representation for the braid group action on exceptional sequences.
- We have a somewhat better understanding when exceptional sequences are strong and when they form helices.

Open questions

- What is the connection to deformation theory in general?
- T-singularities are somewhat distinguished. This should also be visible from the point of representation theory and non-commutative resolutions.
- Has Hackings's construction of exceptional bundles a natural representation-theoretic interpretation?