Formal degrees of unipotent representations

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Representation Theory
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Outline

A conjecture of Hiraga, Ichino and Ikeda (JAMS, 2008) expresses the formal degree of a discrete series representation of the group of points of a connected, semisimple algebraic group over a local field, in terms of local L-functions (assuming the conjectured local Langlands conjecture).

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- Our proof consists of the computation of both sides, the "Hecke side" and the "L-function side". Hecke side: New concepts for affine Hecke algebras (partial ordering, spectral transfer, deformation theory).
- A conjecture (with Dan Ciubotaru): Formal degree = nonabelian FT $\times$ elliptic fake degree vector.

The Weil group of a nonarchimedean local field

- Let k be a nonarchimedian local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((x))$), with ring of integers $\mathfrak{O}$ ($\mathbb{Z}_p$ and $\mathbb{F}_p[[x]]$ respectively), max. ideal $\mathfrak{p} = \pi\mathfrak{O} \subset \mathfrak{O}$, and residue field $\mathfrak{f} = \mathfrak{O}/\mathfrak{p}$ ($= \mathbb{F}_p$ here).

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- Choose $\mathrm{Fr} \in \mathcal{W}_k$, an inverse image of the Frobenius generator of $\mathbb{Z} \subset \mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}) = \hat{\mathbb{Z}}$ (Frobenius endomorphism).

Reductive groups over k, and L-groups

G a connected, reductive k-group, split over a fin. Galois ext. k_0/k ($k \subset k_0 \subset k^s$). Fix Borel $B \subset G$, and max. torus $T \subset B$.

- $\Gamma_{k_0/k} \to \mathrm{Aut}_{abs}(G)$ yields $\Gamma_{k_0/k} \to \mathrm{Aut}((Y, F_0^\vee, X, F_0))$, where $(Y, F_0^\vee, X, F_0)$ is the based root datum of (G, B, T).

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- Two k-rational forms on G defining the same L-group are in the same inner class. The classes of rational forms in an inner class of G are parameterized by $u \in H^1(\Gamma_k, G_{ad})$.
- The inner class of G contains a unique quasi-split form, i.e. a rational form G_0 admitting a k-Borel.

A Theorem of Kottwitz

- Let $Z^\vee \subset G^\vee$ denote the center, and let ${}^L Z = \{e\} \times (Z^\vee)^{\Gamma_k} \subset {}^L G$ (this is the center of $\Gamma_k \ltimes G^\vee$).

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- In terms of ${}^LG$ this assumption is equivalent to: ${}^LZ$ is finite.
- Kottwitz's theorem: There is a canonical bijection $\mathrm{Hom}({}^LZ, \mathbb{C}^\times) \approx H^1(\Gamma_k, G)$ such that the quasi split form G_0 of G corresponds to triv $\in \mathrm{Hom}({}^LZ_{ad}, \mathbb{C}^\times)$. Hence we have a canonical bijection between inner forms of G, and irreducible characters of ${}^LZ_{ad}$.

Local Langlands parameters

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$\phi : \mathcal{WD}_k \to {}^L G = \Gamma_{k_0/k} \ltimes G^\vee$ s.t. (recall $\mathcal{WD}_k = \mathcal{W}_k \times \mathrm{SL}_2(\mathbb{C})$):

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- Observe that ${}^LZ \subset A_\phi$, so discrete ϕ only exist if $|{}^LZ| < \infty$.
- Easy to see: $|A_\phi| < \infty \iff \mathrm{Im}(\phi)$ not contained in proper Levi subgroup. In particular, $(G^\vee)^{\phi(\mathcal{W}_k)}$ is semisimple, and $|\phi(\mathcal{W}_k)| < \infty$.

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- Let $\phi : \mathcal{WD}_k \to {}^L G$ be a discrete Langlands parameter. Let $\mathrm{Irr}(\mathcal{A}_\phi, \zeta_G)$ be the set of cpx. irreps. of $\mathcal{A}_\phi$ which restrict to $\zeta_G.\mathrm{Id}$ on $Z_{ad}^{\vee}$.

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- Call a pair (ϕ, ρ), with $\phi : \mathcal{WD}_k \to {}^L G$ a discrete Langlands parameter (DLP), and $\rho \in \mathrm{Irr}(\mathcal{A}_\phi, \zeta_G)$ an enhanced DLP for G (abbreviated by EDLP).

The conjectured local Langlands correspondence for the discrete series

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There exists a bijection LLC

$$\{\pi \mid \pi \text{ irred. D.S. of G(k)}\}/_\sim \xrightarrow{\sim} \{(\phi,\rho) \mid (\phi,\rho) \text{ EDLP for G }\}/_\sim$$

- We will write (ϕ_π, ρ_π) for the EDLP corresponding to a discrete series representation π of $G(k)$, and $\pi_{(\phi,\rho)}$ for the discrete series representation of $G(k)$ corresponding to an EDLP (ϕ, ρ) for G.

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- We call $\Pi_\phi := \sqcup_{u\in H^1(\Gamma_k,(G_0)_{ad}))}\Pi_\phi^u$ the L-packet of ϕ.

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Among many others, the correspondence is expected to have the following properties.

- Unipotent representations. Assume $G = G_{ad}$. Then π is unipotent in the sense of Lusztig if and only if ϕ_π is unramified (i.e. is trivial on the inertia group I). In this case, the LLC should be the bijection Lusztig has constructed via arithmetic-geometric correspondences for Hecke algebras.

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- All elements of Π_ϕ are supercuspidal if and only if the generic member is supercuspidal if and only if $\phi|_{\mathsf{SL}_2} = \text{triv}$.

The conjectured local Langlands correspondence for the discrete series

Among many others, the correspondence is expected to have the following properties.

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- Stability properties of certain weighted sums of characters in an L-packet.

The local L function

Let $\phi : \mathcal{WD}_k \to {}^L G$ be a Langlands parameter for k, and assume that (V, r) is a f.d. complex representation of ${}^L G$. Then $(V, r \circ \phi)$ is a f.d. cpx. representation of $\mathcal{WD}_k$.

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Let $\phi : \mathcal{WD}_k \to {}^LG$ be a Langlands parameter for k, and assume that (V, r) is a f.d. complex representation of ${}^LG$. Then $(V, r \circ \phi)$ is a f.d. cpx. representation of $\mathcal{WD}_k$. So we have

$$V \equiv \oplus_{n \geq 0} V_n \otimes \mathrm{Sym}^n$$

where Sym^n is the $n+1$-dimensional irrep. of $\mathrm{SL}_2(\mathbb{C})$, and V_n a f.d. cpx. repr. of $\mathcal{W}_k$.

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Definition of the local L-function

$L(\phi, V, s) := \prod_{n \geq 0} \det(1 - v^{-n-2s} r(\phi(\mathrm{Fr}))|_{V_N^{\mathrm{I}}})^{-1}$

γ-factors and discrete Langlands parameters

Definition of γ-factor

For $s \in \mathbb{C}$ define $\gamma(\phi, V, s) := \frac{L(\phi, V^*, 1-s)\epsilon(\phi, V, s)}{L(\phi, V, s)}$, where $\epsilon(\phi, V, s)$ is the local ϵ factor (a root of 1 times an integral power of v^{1-2s}).

- Proposition (Gross, Reeder): ϕ is discrete $\iff L(\phi, \mathrm{Ad}, s)$ is regular in $s = 0$ (where Ad is the adjoint representation of ${}^LG$ on $\mathfrak{g}^\vee$).

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- Corollary ϕ is discrete if and only if the adjoint γ factor $\gamma(\phi, \mathrm{Ad}, 0)$ is nonzero.

Normalization of measures, and formal degrees

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- Put $\overline{\mathbb{P}} := \mathbb{P}/\mathbb{U}_{\mathbb{P}}$. We normalize Haar measure μ of $G(k)$ such that: $\mathrm{Vol}(\mathbb{P}) = v^{-\dim(\overline{\mathbb{P}})}|\overline{\mathbb{P}}|$ for all parahorics $\mathbb{P} \subset G(k)$.

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- A discrete series representation (V, π) of $G(k)$ has (by definition) square integrable matrix coefficients $c_{\phi,v}(g) := \phi(\pi(g)v)$ (for $\phi \in V^*$ and $v \in V$). There exists a constant $\mathrm{fdeg}(\pi) > 0$ s.t. $\mathrm{fdeg}(\pi)\|c_{\phi,v}\|^2_{L^2(G(k),\mu)} = |\phi(v)|^2$.

The Hiraga-Ichino-Ikeda-conjecture

Conjecture of Hiraga, Ichino and Ikeda (2006)

Assume k is a nonarchimedian local field, and G a connected reductive over k, with Haar measure normalized as above. Assume the local LLC $\pi \to (\phi_\pi, \rho_\pi)$ for G. Let π be a discrete series representation of $G(k)$. Then

$$\mathrm{fdeg}(\pi) = \pm \frac{\dim(\rho_\pi)}{|A_{\phi_\pi}/{}^L Z|} v^{-\dim(G)} \gamma(\phi_\pi, \mathrm{Ad}, 0)$$

Remark. The conjecture would imply that the formal degree $\mathrm{fdeg}(\pi)$ with this normalization of Haar measures has a unique decomposition $\mathrm{fdeg}(\pi) = \mathrm{fdeg}_{\mathbb{Q}}(\pi)\mathrm{fdeg}_q(\pi)$ where $\mathrm{fdeg}_{\mathbb{Q}}(\pi)$ is a rational number, and $\mathrm{fdeg}_q(\pi)$ a fraction of products of q-rational numbers of the form $\frac{v^n - v^{-n}}{v - v^{-1}}$.

Known results

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- Lusztig calls a smooth irrep π of $G(k)$ unipotent if there exists a parahoric $\mathbb{P}$ of $G(k)$, and a cuspidal unipotent representation σ of $\overline{\mathbb{P}}$ (a reductive finite group of Lie type), such that σ appears in $\pi|_{\mathbb{P}}$. Reeder (2000) proved it for unipotent discrete series of split exceptional groups.

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- Waldspurger, Moeglin proved stability of the appropriate sums of unipotent characters, for some classical groups like SO_{2n+1}, SU_n,... Using the fundamental Lemma, and Arthur's work: The HII-conjecture follows for unipotent representations of these groups.

Main Theorems

Theorem (O., 2014)

If we accept Lusztig's parametrization of unipotent representations of an unramified semisimple adjoint type group G over k as the LLC, then the conjecture of H-I-I holds for all unipotent L-discrete series L-packets.

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Uniqueness Theorem (Y. Feng-O., 2014)

Assume G is simple. Up to tensoring by unramified characters of $G(k)$, for unipotent discrete series representations π of $G(k)$, the discrete Langlands parameter ϕ_π is completely determined by $\mathrm{fdeg}_q(\pi)$ (assuming the conjecture of H-I-I).

Unipotent affine Hecke algebras

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- This means that:
  $\mathcal{R}(G(k))_t := \{\pi \text{ smooth } G(k) - \text{rep} \mid V_\pi = \mathcal{H}(G(k))V_\pi^\sigma\}$ is an abelian subcategory of the category of smooth representations of $G(k)$.

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- Then $\mathcal{R}(G(k))_t \xrightarrow{\sim} \mathcal{H}_q^{u,t} - \text{mod}$, by $V \to V^\sigma := \text{Hom}_{\mathbb{P}}(\sigma, V)$ where $\mathcal{H}_q^{u,t} = \text{End}_{G(k)}(\text{cInd}_{\mathbb{P}}^{G(k)}(\sigma))^{op}$.

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- $\mathcal{H}_q^{u,t}$ is an extended affine Hecke algebra.

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- Why can we be so precise in the Plancherel formula for affine Hecke algebras? The key points are: Deformation with respect to Hecke parameters (O.,O.-Solleveld (q-rat. fact.)); and Dirac induction AHA. (Ciubotaru-Kato, Trapa-Ciubotaru-O., Ciubotaru-O. ($\mathbb{Q}$-fact.)). Both techniques are not available for p-adic groups themselves.

Plancherel measure and Langlands parameters

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- The answer lies in the relation between Lusztig's "arithmetic-geometric" correspondences in terms of the so called μ-function μ^{qs} of the Iwahori Hecke algebra of G_0.
- To explain this in a bit more detail, we need to introduce some notations for affine Hecke algebras.

Affine Hecke algebras

- An affine Hecke algebra $\mathcal{H}_q$ over $\mathbb{C}$ is a deformation of the complex group algebra $\mathbb{C}[W]$ of an extended affine Weyl group $W = X \rtimes W_0$ with W_0 the Weyl group of a based root datum $(X, F_0, Y, F_0^\vee)$, according to certain Hecke parameters $q_i = v^{n_i}$ associated with the simple reflections $s_0, s_1, \ldots, s_n$.

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- $\mathcal{H}_q$ is associative and unital, and it has a distinguished $\mathbb{C}$-basis $\{N_w\}_{w \in W}$ (with $w \in W$) such that $N_{w_1} N_{w_2} = N_{w_1 w_2}$ (if $l(w_1 w_2) = l(w_1) + l(w_2)$) and $(N_{s_i} - v_i)(N_{s_i} + v_i^{-1}) = 0$.

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- There exists a unique abelian subalgebra $\mathcal{H} \supset A \simeq \mathbb{C}[X]$ with a distinguished $\mathbb{C}$-basis $\{\theta_x\}_{x \in X}$ such that $\theta_x = N_x$ for $x \in X^+$, and $\theta_x \theta_y = \theta_{x+y}$ for all $x, y \in X$ (Bernstein basis). As vector spaces: $A \otimes \mathcal{H}_0 \xrightarrow{\sim} \mathcal{H}$ and $\mathcal{H}_0 \otimes A \xrightarrow{\sim} \mathcal{H}$ (via the multiplication maps).

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- The pair (T, μ) determines the pair $(\mathcal{H}, \tau)$ (normalized AHA).

Spectral transfer maps (STM)

Definition

A spectral transfer morphism $\phi : (\mathcal{H}_1, \tau_1) \rightsquigarrow (\mathcal{H}_2, \tau_2)$ is a $W_{2,0}$-orbit $W_{2,0}\phi_T$ of affine morphisms $\phi_T : T_1 \to T_2$ of algebraic tori satisfying

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(T3) $\exists\, a \in \mathbb{Q}^\times$ such that $\phi_T^*(\mu_2^{(L)}) = a\mu_1$, where $\mu_2^{(L)}$ denotes a regularization of μ_2 along L.

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(T2) $\mathrm{Im}(\phi_T) := L \subset T_2$ is a coset of a subtorus along which μ_2 has maximal pole order.

(T3) $\exists\, a \in \mathbb{Q}^\times$ such that $\phi_T^*(\mu_2^{(L)}) = a\mu_1$, where $\mu_2^{(L)}$ denotes a regularization of μ_2 along L.

Theorem (rough version) Given STM $\phi : (\mathcal{H}_1, \tau_1) \rightsquigarrow (\mathcal{H}_2, \tau_2)$, the Pl. measure $\nu_{Pl,1}$ of $(\mathcal{H}_1, \tau_1)$ is closely related to restriction of $\nu_{Pl,2}$ to the components in the tempered spectrum of $\mathcal{H}_2$ which map to $W_0 L$ under the central character map.

Lusztig's arithmetic-geometric corresp. and μ^{qs}

G_0 a unramified quasi split group over k. Let $\mathcal{H}^{qs} = \mathcal{H}^{IM}(G_0)$ be the Iwahori Hecke algebra of G_0, with μ-function μ^{qs}.

- Lusztig: $\oplus_{(u,t)}\mathcal{H}^{u,t} \xrightarrow{\sim} \oplus\mathcal{H}(G^\vee, H^\vee, \mathcal{C}, \mathcal{F})$, where the sum on the RHS is over the set of Int$(G^\vee)$-orbits of triples $(H^\vee, \mathcal{C}, \mathcal{F})$ with $H^\vee \subset G^\vee$ reductive, $\mathcal{C} \subset H^\vee$ dist. unip. orbit, and F a cusp. local cyst. on $\mathcal{C}$. Here $\mathcal{H}(G^\vee, H^\vee, \mathcal{C}, \mathcal{F})$ is an affine Hecke algebra which Lusztig attaches to the data $(G^\vee, H^\vee, \mathcal{C}, \mathcal{F})$.

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- Theorem O., Feng-O. For all (u, t) we have a bijection: $\{\phi : \mathcal{H}_{u,t} \rightsquigarrow \mathcal{H}^{qs}\} \xrightarrow{\sim} \{(H^\vee, \mathcal{C}, \mathcal{F}) \mid \mathcal{H}(G^\vee, H^\vee, \mathcal{C}, \mathcal{F}) \simeq \mathcal{H}_{(u,t)}\}$.

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- Corollary. The conjecture of H-I-I holds for Lusztig's parameters, at least for the q-rational factors.

Partial ordering on AHAs; uniqueness results

- We say $(\mathcal{H}_1, \tau_1) \sim (H_2, \tau_2)$ (spectral isogeny) if there exist STMs $\phi : (\mathcal{H}_1, \tau_1) \rightsquigarrow (\mathcal{H}_2, \tau_2)$ and $\psi : (\mathcal{H}_2, \tau_2) \rightsquigarrow (\mathcal{H}_1, \tau_1)$.

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- (Nondegenerate case): Spectral isogeny implies $\exists \alpha : \mathcal{H}_1 \xrightarrow{\sim} \mathcal{H}_2$ (as AHAs) and $\alpha^*(\tau_2) = c\tau_1$ with $c \in \mathbb{Q}^\times$.

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- Theorem: Let G_0 be unramified, quasi split group over k, with Iwahori Hecke algebra $\mathcal{H}^{qs}$ as before. In the set of all normalized AHAs $\{\mathcal{H}^{u,t}\}$ arising form unipotent types t of inner forms $G = G_0^u$ of G_0, $\mathcal{H}^{qs}$ is the lowest element.

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- This recovers Lusztig's classification, in terms of μ^{qs} and the classification of cuspidal unipotent representations of finite groups of Lie type, and yields the H-I-I conjecture.

Elliptic fake degrees and formal degrees

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- For all $s \in A_u := C_{G^\vee}(\phi_u)$, let $\phi_{(s,u)} : \mathcal{W}_k \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ be the DLP such that $\phi_{(s,u)}|_{\mathrm{SL}_2(\mathbb{C})} = \phi_u$ and $\phi_{(s,u)}(\mathrm{Fr}) = s$.

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- Let Σ_u be the set of equiv. classes of EDLP (ϕ, σ) with $\phi = \phi_{(s,u)}$ for some $s \in A_u$. Then $\Sigma_u/_\sim \xrightarrow{\sim} \{(s,\tau) \mid s \in A_u,\ \tau \in \mathrm{Irr}(C_{A_u}(s))\}/_\sim := M(A_u)$, the set of irreps of the Drinfeld double $D(A_u)$.

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- Let S be the "S-matrix" of the Verlinde algebra $K(D(A_u) - \mathrm{mod})$ (Lusztig's nonabelian Fourier transform).

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- Conjecture (w. Ciubotaru) Let $\pi \in \mathcal{R}_{unip}(G(k))_{DS}$ such that $(\phi_\pi, \rho_\pi) \in \Sigma_u \leftrightarrow (s, \tau) \in M(A_u)$. Then we have: $\mathrm{fdeg}(\pi) = \frac{1}{|{}^L Z|} \sum_{(s',\tau')} S_{(s,\tau),(s',\tau')} F^{ell}_{\lim_{q\to 1}(\pi')^{\mathbb{B}}}$, where for $\rho \in \mathrm{Rep}(W_0)$ we put $F^{ell}_{\rho} := \frac{(q-1)^n}{|W_0|} \sum_{w \in W_0} \chi_\rho(w) \frac{\det(1-w)}{\det(1-qw)}$ (the elliptic fake degree of ρ).

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- We checked it for small rank cases (including all exceptional groups).

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