Partially dualized Hopf algebras have equivalent Yetter-Drinfel'd modules

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joint work with Alexander Barvels & Christoph Schweigert
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Overview

Definitions: Braided Categories

Let $\mathcal{C}$ be an abelian category (direct sums, products, kernel, cokernel etc.) We denote objects by $X, Y, \ldots$ and morphism spaces by $\mathrm{Hom}_{\mathcal{C}}(X, Y), \ldots$

- A **monoidal** category $\mathcal{C}$ is a category with a tensor product $X \otimes Y \in \mathcal{C}$, a natural associativity isomorphism, unit object...
- A **braided** category $\mathcal{C}$ is a monoidal category with a natural commutativity isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$.

The structural data is subject to several **coherence axioms**.

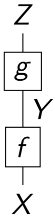
Example

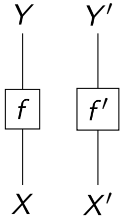
The following are examples of braided categories:

- The category of vector spaces over a field.
- The category of left modules over a Hopf algebra with R-matrix.
- The category of Yetter-Drinfel'd modules over a group (later).

Definitions: Graphical Calculus

We visualize morphisms in a braided category $\mathcal{C}$ by diagrams. Examples:

$g \circ f =$ 

$f \otimes f' =$ 

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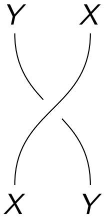
$c_{X,Y} =$ 

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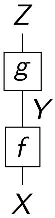
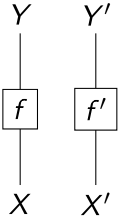
$g \circ f =$ 

$f \otimes f' =$ 

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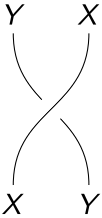
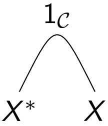
$c_{X,Y} =$ 

A **dual** to an object X is an object X^* and morphisms

Eval $=$ 

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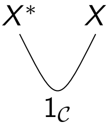
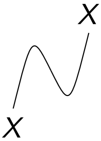
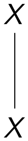
Coeval $=$ 

with 

$=$ 

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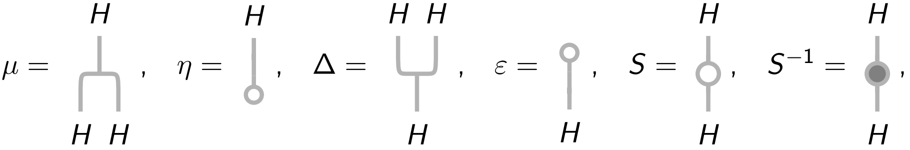
Definitions: Hopf algebras

A Hopf algebra in a braided category $\mathcal{C}$ is an object $H \in \mathcal{C}$ with morphisms



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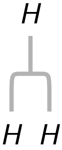
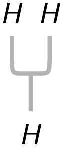
$\mu =$ 

, $\eta =$ 

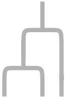
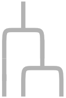
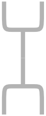
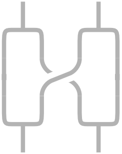
, $\Delta =$ 

, $\varepsilon =$ 

, $S =$ 

, $S^{-1} =$ 

,

μ is **associative**  $=$  , Δ is **multiplicative**  $=$  etc.

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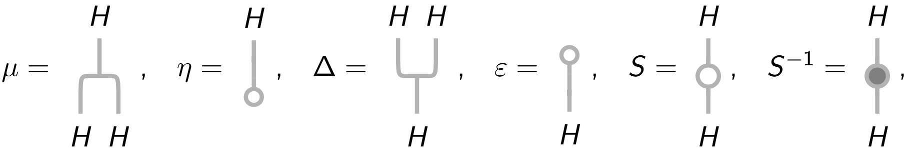


μ is **associative** , Δ is **multiplicative** etc.

Theorem

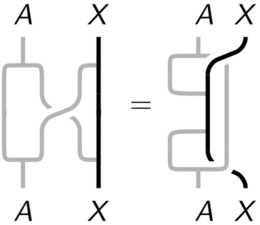
The category of representations of a Hopf algebra $H \in \mathcal{C}$ and the tensor product $V \otimes_{\mathcal{C}} W$ with H-action via Δ is a **monoidal category** $_H\mathrm{Mod}(\mathcal{C})$.

Definitions: Yetter-Drinfel'd modules

Let $A \in \mathcal{C}$ be a Hopf algebra with bijective antipode in a braided category.

A **Yetter-Drinfel'd module** X over H in $\mathcal{C}$ is:

- An object $X \in \mathcal{C}$
- with left H-module- and H-comodule structure
- fulfilling the Yetter-Drinfel'd condition:



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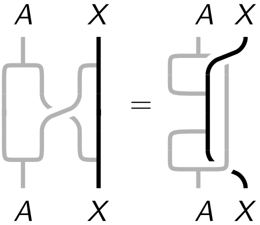

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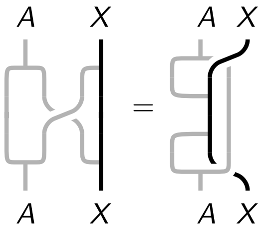

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- *The category of H-Yetter-Drinfel'd modules ${}^H_H\mathcal{YD}(\mathcal{C})$ over a Hopf algebra $H \in \mathcal{C}$ with bijective antipodes is a* **braided category**.
- *It is equivalent to the category of modules over a Hopf algebra with R-matrix in $\mathcal{C}$, the so-called* **Drinfel'd double** $\mathcal{D}(H) \in \mathcal{C}$.

Partial Dualization: Step 1

Let $H \in \mathcal{C}$ be a Hopf algebra with a Hopf algebra projection to a Hopf subalgebra in $\mathcal{C}$

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Theorem (Radford projection theorem, Part 1)

The space of π-coinvariants $K := H^{\mathrm{coin}(\pi)}$ is a Hopf algebra K **in the braided category** ${}^{A}_{A}\mathcal{YD}(\mathcal{C})$.

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Explicitly, the Hopf algebra K is given by

$$K := \left\{ h \in H \mid h^{(1)} \otimes \pi(h^{(2)}) = h \otimes 1 \right\}$$

with the following product and coproduct in ${}^A_A\mathcal{YD}(\mathcal{C})$

$$\mu_K(h \otimes h') := \mu_H(h \otimes h')$$

$$\Delta_K(h) := h^{(1)}\pi(S_H(h^{(2)})) \otimes h^{(3)}$$

Partial Dualization: Step 2

Let A be a Hopf algebra in $\mathcal{C}$ with dual A^*.
The Yetter-Drinfel'd categories over A and A^* are braided equivalent:

$$\Omega : \ {}^{A}_{A}\mathcal{YD}(\mathcal{C}) \xrightarrow{\cong} {}^{A^*}_{A^*}\mathcal{YD}(\mathcal{C})$$
$$K \longmapsto L$$

This defines a Hopf algebra $L \in {}^{A^*}_{A^*}\mathcal{YD}(\mathcal{C})$.

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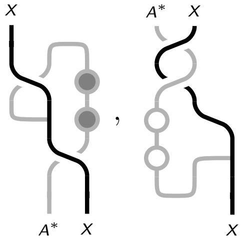
Diagrammatically, the A^*-action and A^*-coaction for the Yetter-Drinfel'd module $L := \Omega(K)$ are given as follows:

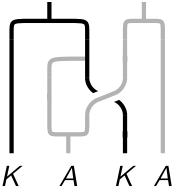


Let $A \in \mathcal{C}$ be a Hopf algebra and $K \in {}^A_A\mathcal{YD}(\mathcal{C})$ be a Hopf algebra, then

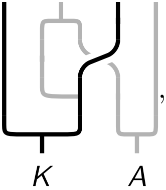
Definition

The **Radford biproduct** $K \rtimes A$ *is the object* $K \otimes A \in \mathcal{C}$.
The following structures turn the object into a Hopf algebra in $\mathcal{C}$.

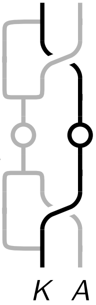
$\mu_{K\rtimes A} :=$



, $\Delta_{K\rtimes A} :=$



, $S_{K\rtimes A}$



Partial Dualization: Step 3

Let $H \in \mathcal{C}$ be a Hopf algebra with projection $\pi : H \underset{\hookleftarrow}{\longrightarrow} A$

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There is an isomorphism of Hopf algebras in $\mathcal{C}$:

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Definition

The **partially dualized Hopf algebra** *with respect to π is defined as*

$$r(H) := L \rtimes A^* \qquad \text{with} \quad L := \Omega(K) \in {}^{A^*}_{A^*}\mathcal{YD}(\mathcal{C})$$

and is a Hopf algebra in $\mathcal{C}$ with projection $\pi' : r(H) \underset{\hookleftarrow}{\longrightarrow} A^*$

Obvious Questions

- Iteration: $H \mapsto r(H) \mapsto r(r(H)) \mapsto \cdots$

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  - In general neither isomorphic nor Morita equivalent
  - For Nichols algebras: so-called Weyl equivalence

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  $\Rightarrow$ Can we clarify their relation?

Representation theoretic implications

In general, $H, r(H)$ have non-equivalent categories of (co)modules, **but:**

Theorem

The categories of Yetter-Drinfel'd modules are braided equivalent:

$$ {}^{H}_{H}\mathcal{YD}(\mathcal{C}) \;\cong\; {}^{r(H)}_{r(H)}\mathcal{YD}(\mathcal{C}) $$

The proof uses an isomorphism ${}^{K}_{K}\mathcal{YD}({}^{A}_{A}\mathcal{YD}(\mathcal{C})) \cong {}^{L}_{L}\mathcal{YD}({}^{A^*}_{A^*}\mathcal{YD}(\mathcal{C}))$.

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Corollary

- *The Drinfel'd doubles $\mathcal{D}(H), \mathcal{D}(r(H))$ are Morita equivalent algebras.*
- *As ${}_H\mathrm{Mod}(\mathcal{C})$ is a monoidal category, it is natural to consider its* **module categories** $\mathcal{M}$, *which form a bicategory ${}_H\mathrm{Mod}(\mathcal{C}) - \mathcal{MOD}$. If H is semisimple, we can use existing results to see:*

$$ {}_H\mathrm{Mod}(\mathcal{C}) - \mathcal{MOD} \cong {}_{r(H)}\mathrm{Mod}(\mathcal{C}) - \mathcal{MOD} $$

Examples of partial dualization

- **Extreme cases** are:

$$H \underset{\hookleftarrow}{\longrightarrow} 1 \qquad r(H) = H \underset{\hookleftarrow}{\longrightarrow} 1$$

$$H \underset{\hookleftarrow}{\longrightarrow} H \qquad r(H) = H^* \underset{\hookleftarrow}{\longrightarrow} H^*$$

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- A **group algebra** $H = k[N \rtimes Q]$ has as partial dualization:

$$H = k[N \rtimes Q] \underset{\hookleftarrow}{\longrightarrow} k[Q] \qquad r(H) = k[N] \rtimes k^Q \underset{\hookleftarrow}{\longrightarrow} k^Q$$

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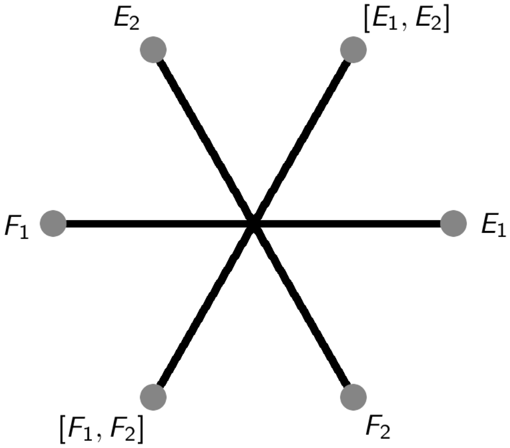
- Motivating notion is the **reflection**
  - in quantum groups $u_q(\mathfrak{g})$.
  - of Nichols algebras (Andruskiewitsch, Heckenberger, Schneider)

Theorem

Partial dualization describes the image of a quantum Borel part $u_q(\mathfrak{g})^+$ or a Nichols algebra under a simple reflection as Hopf algebra.

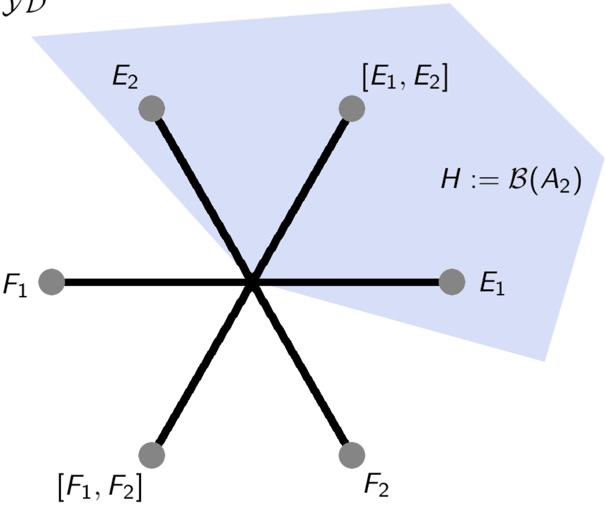
Examples: Reflection in quantum groups

$U_q(\mathfrak{sl}_3)$



Examples: Reflection in quantum groups

$\mathcal{C} := {}^{\mathbb{Z}^2}_{\mathbb{Z}^2}\mathcal{YD}$



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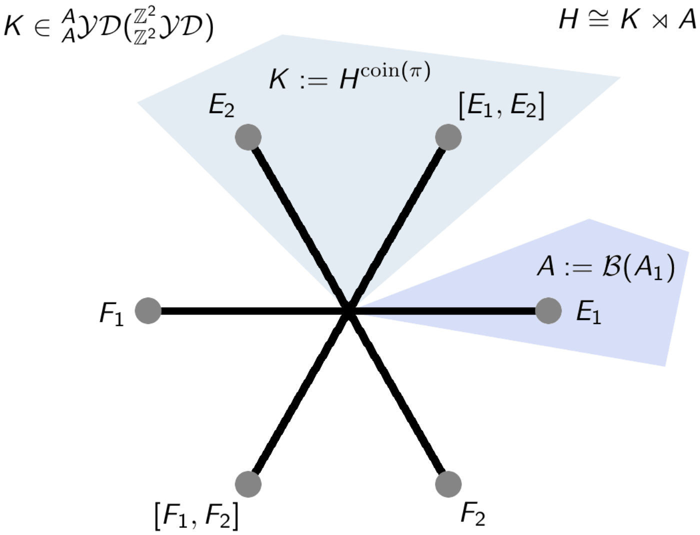
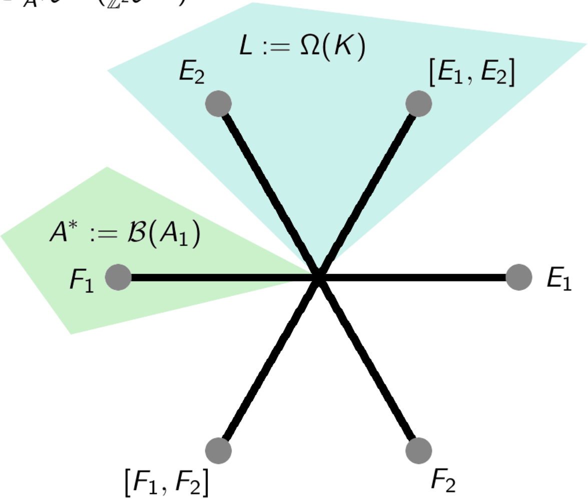


Examples: Reflection in quantum groups

$L \in {}^{A^*}_{A^*}\mathcal{YD}({}^{\mathbb{Z}^2}_{\mathbb{Z}^2}\mathcal{YD})$



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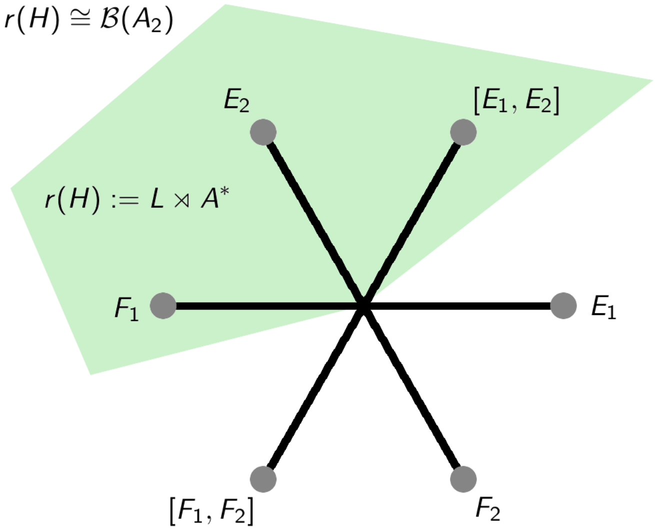

OVERVIEW TRANSPARENCY

Let $\mathcal{C}$ be a braided abelian category, e.g. vector spaces.

Let $H \in \mathcal{C}$ be a Hopf algebra with a Hopf algebra projection to a Hopf subalgebra $A \in \mathcal{C}$, which has a dual $A^* \in \mathcal{C}$

$$\pi : H \underset{\hookleftarrow}{\longrightarrow} A$$

We construct the **partially dualized** Hopf algebra $r(H) \in \mathcal{C}$:

$$\mathcal{C} \dashrightarrow {}^{A}_{A}\mathcal{YD}(\mathcal{C}) \xrightarrow{\cong} {}^{A^*}_{A^*}\mathcal{YD}(\mathcal{C}) \dashrightarrow \mathcal{C}$$

$$H \xmapsto{\text{Step 1}} K \xmapsto{\text{Step 2}} L \xmapsto{\text{Step 3}} r(H)$$

(Formulae for Picture)

400%

$$E_1 \qquad E_2 \qquad F_1 \qquad F_2 \qquad [E_1, E_2] \qquad [F_1, F_2]$$

$$U_q(\mathfrak{sl}_3)$$

$$H := \mathcal{B}(A_2)$$

$$\mathcal{C} := {}^{\mathbb{Z}^2}_{\mathbb{Z}^2}\mathcal{YD}$$

$$H \cong K \rtimes A$$

$$K := H^{\mathrm{coin}(\pi)}$$

$$K \in {}^{A}_{A}\mathcal{YD}({}^{\mathbb{Z}^2}_{\mathbb{Z}^2}\mathcal{YD})$$

$$A := \mathcal{B}(A_1)$$

$$L := \Omega(K)$$

$$L \in {}^{A^*}_{A^*}\mathcal{YD}({}^{\mathbb{Z}^2}_{\mathbb{Z}^2}\mathcal{YD})$$

$$A^* := \mathcal{B}(A_1)$$

$$r(H) := L \rtimes A^*$$