Cluster structures on strata of flag varieties

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Dedicated to Andrei Zelevinsky (1953–2013)

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He introduced nice stratifications of $G_{\geq 0}$ and $(P\backslash G)_{\geq 0}$.

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Definition (Fomin-Zelevinsky 1999)

For $v, w \in W$, let $G^{v,w} := (B^- vB^-) \cap (B^+ wB^+)$. We have:

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Theorem (Fomin-Zelevinsky 1999)

- $G^{v,w}$ is a smooth irreducible quasi-affine variety over $\mathbb{C}$ of dimension $\ell(v) + \ell(w) + r$.
- $G^{v,w} \cap G_{\geq 0}$ is homeomorphic to $\mathbb{R}_{>0}^{\ell(v)+\ell(w)+r}$.

WHAT IS...

a Cluster Algebra?

Andrei Zelevinsky

Cluster algebras, first introduced in [2], are constructively defined commutative rings equipped with a distinguished set of generators *(cluster variables)* grouped into overlapping subsets *(clusters)* of the same finite cardinality (the *rank* of an algebra in question). Among these algebras one finds coordinate rings of many algebraic varieties that play a prominent role in representation theory, invariant theory, the study of total positivity, etc. For instance, homogeneous coordinate rings of Grassmannians, Schubert varieties, and other related varieties carry a cluster algebra structure (after a minor adjustment). Potential applications of this structure include explicit constructions of the (dual) canonical basis and toric degenerations for these varieties.

Since its inception, the theory of cluster algebras has found a number of exciting connections and applications: quiver representations, preprojective algebras, Calabi-Yau algebras and categories, Teichmüller theory, discrete integrable systems, Poisson geometry... The current state of these developments, including links to papers, working seminars, conferences, etc., is represented at the online Cluster Algebras Portal created and maintained by S. Fomin [1].

Although some of the above connections are rather technical, cluster algebras themselves are defined in an elementary manner not requiring any tools beyond high-school algebra. On the other hand, they have an unusual feature that both generators and algebraic relations among them are not given from the outset but are produced by an iterative process of *seed mutations.*

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Before discussing the general definition, let us look at cluster algebras of rank two. One associates such an algebra $\mathcal{A}(b,c)$ with any pair (b,c) of positive integers. The cluster variables in $\mathcal{A}(b,c)$ are the elements x_m, for $m \in \mathbb{Z}$, defined recursively by the *exchange relations*

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1 & \text{if } m \text{ is odd;} \\ x_m^c + 1 & \text{if } m \text{ is even.} \end{cases}$$

Iterating these relations, we can express each x_m as a rational function of x_1 and x_2. Thus, $\mathcal{A}(b,c)$ is the subring generated by all the x_m inside the field of rational functions $\mathbb{Q}(x_1,x_2)$. The clusters are the pairs $\{x_m, x_{m+1}\}$ for $m \in \mathbb{Z}$. Starting with the initial cluster $\{x_1,x_2\}$, we can reach any other cluster by a series of exchanges

$$\cdots \leftrightarrow \{x_0,x_1\} \leftrightarrow \{x_1,x_2\} \leftrightarrow \{x_2,x_3\} \leftrightarrow \ldots .$$

For an arbitrary rank n, the construction is similar. Each cluster $\mathbf{x} = \{x_1,\ldots,x_n\}$ is a collection of algebraically independent elements of some ambient field, and each cluster variable x_k can be exchanged from $\mathbf{x}$ by forming a new cluster $\mathbf{x}' = \mathbf{x} - \{x_k\} \cup \{x_k'\}$. Here x_k and x_k' are related by an exchange relation of the following form: the product $x_k x_k'$ is equal to the sum of two disjoint monomials in the variables from $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} - \{x_k\}$. (For simplicity, we restrict ourselves to the *coefficient-free* case, where both monomials appear with the coefficient 1.) The exponents in these two monomials are encoded by an $n \times n$ integer matrix $B = (b_{ij})$ called the *exchange matrix*; it is usually assumed to be skew-symmetrizable, that is, $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1,\ldots,d_n$. The corresponding exchange relations take the form

$$x_k x_k' = \prod_i x_i^{[b_{ik}]_+} + \prod_i x_i^{[-b_{ik}]_+},$$

where we use the notation $[b]_+ = \max(b,0)$.

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- $\mathbb{C}[G^{v,w}]$ has a cluster structure compatible with $G^{v,w} \cap G_{\geq 0}$.
- Every cluster gives an optimal total positivity criterion.

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- What about $(P\backslash G)_{\geq 0}$?

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Proposition (Kazhdan-Lusztig, Deodhar)

- The intersection $\mathscr{R}_{v,w} := C^v \cap C_w$ is nonempty iff $v \leq w$.
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  $\mathscr{R}^P_{v,w} =$ "open positroid variety" (Postnikov, Knutson-Lam-Speyer).

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Aim

Describe a "categorical model" for a cluster structure on $\mathscr{R}_{v,w}$ for any simply-laced group G.

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- Mutation $T \rightsquigarrow T'$ induces Fomin-Zelevinsky quiver mutation $\Gamma_T \rightsquigarrow \Gamma'_T$.

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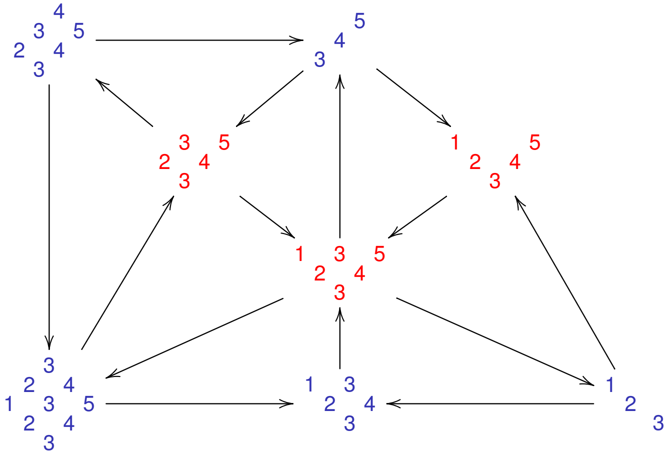
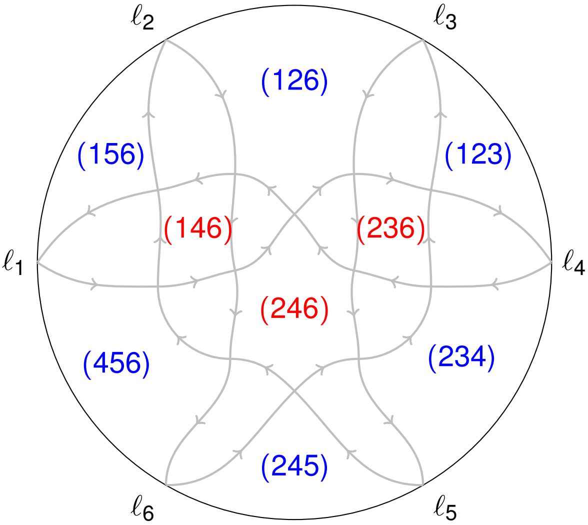
- For positroid varieties the theorem agrees with the Muller-Speyer conjecture.

Type A_5 $\quad v = s_1 s_2 s_1 s_4 s_5 s_4 \quad w = w_0 s_2$



$\mathrm{Gr}(3,6) \quad [345] = 0$



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