Finite dimensional representations of rational Cherednik algebras

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$\mathbb{C}[V]$, G, and the Dunkl operators $y(\cdot)$ (for any $y \in V$)

$$y(f) = \partial_y(f) - \sum_{s \in R} c_s \langle \alpha_s, y \rangle \frac{f - s(f)}{\alpha_s} \quad \text{for } f \in \mathbb{C}[V].$$

Standard modules

Theorem (PBW for rational Cherednik algebras)

$\mathbb{C}[V] \otimes \mathbb{C}G \otimes \mathbb{C}[V^*] \cong H_c(G, V)$ *as vector spaces.*

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The standard module corresponding to $\lambda \in \Lambda$ is

$$\Delta_c(\lambda) := \mathrm{Ind}_{\mathbb{C}[V^*] \rtimes G}^{H_c(G,V)} S^\lambda$$

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Facts

- $\Delta_c(\lambda)$ has a unique irreducible quotient $L_c(\lambda)$

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knowing the set $\{\lambda \in \Lambda \mid L_c(\lambda) \text{ fin diml}\}$

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Let $M \in H_c(G, V)$-mod (and hence $M \in \mathbb{C}[V]$-mod). The support of M is the set

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Necessary condition on λ and c for $L_c(\lambda)$ to be finite dimensional.

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with relations

$$TgT^{-1} = hgh^{-1} \qquad \text{for } g \in G_S$$

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and the action of T is given by

$$T \cdot x = e^{2\pi i \tau} h(x) \qquad \text{for all } x \in S^\nu_\tau$$

Main theorem

- for $H \subseteq G$, $S^\lambda \in \mathrm{Irr}(\mathbb{C}G\text{-mod})$ and $S^\mu \in \mathrm{Irr}(\mathbb{C}H\text{-mod})$, let $S^{\lambda,\mu}$ denote the S^μ-isotypic component of $\mathrm{res}^G_H S^\lambda$.

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Theorem (Griffeth, Gusenbauer, Juteau, L.)

Let $S = \mathbb{C}^\times v$ for some $v \in V \setminus \{0\}$. If $S \not\subseteq \mathrm{supp}(L_c(\lambda))$, then for each irreducible representation S^ν of N_S with $S^{\lambda,\nu} \neq 0$, there is an irreducible $\mathbb{C}G$-module S^μ and an irreducible $\mathbb{C}N_S$-module S^η such that

(1) $\mu >_c \lambda$,

(2) $S^{\mu,\eta} \neq 0$, *and*

(3) $S^\nu_{(c_\lambda - c_\nu)/n_S} \cong S^\eta_{(c_\mu - c_\eta)/n_S}$ *as B_S-modules.*

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There is a weaker necessary condition that it's much easier to check in practice:

Corollary (Griffeth, Gusenbauer, Juteau, L.)

If $L_c(\lambda)$ is finite dimensional, then for each one-dimensional subspace $S \subseteq V$, the class of $res^G_{G_S}(S^\lambda)$ in $K_0(\mathbb{C}G_S\text{-}mod)$ belongs to the subgroup generated by the classes of $res^G_{G_S}(S^\mu)$ for all $\mu >_c \lambda$.

Example: $B_2 = G(2,1,2)$

Let $G = B_2 = G(2,1,2)$ and c_1, resp. c_2, be the parameter corresponding to the long, resp. short, roots. GAP indexes the irreducibles by $\{11. \, , \, 1.1 \, , \, .11 \, , \, 2. \, , \, .2\}$. If we assume $c_1 > c_2$, the (c-reordered) restriction table to the two (maximal) parabolic subgroups is

		$\sim A_1$		$A_1(2)$	
c^λ	λ	11	2	11	2
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Let $G = B_2 = G(2,1,2)$ and c_1, resp. c_2, be the parameter corresponding to the long, resp. short, roots. GAP indexes the irreducibles by $\{11. \ , \ 1.1 \ , \ .11 \ , \ 2. \ , \ .2\}$. If we assume $c_1 > c_2$, the (c-reordered) restriction table to the two (maximal) parabolic subgroups is

c^λ	λ	$\sim A_1$ 11	2	$A_1(2)$ 11	2
0	2 .	.	1	.	1
$4c_2$	11 .	.	1	1	.
$2c_1 + 2c_2$	1.1	1	1	1	1
$4c_1$	. 2	1	.	.	1
$4c_1 + 4c_2$	. 11	1	.	1	.

(1) $c_1 > c_2 > 0 \Rightarrow L_c(1.1)$, $L_c(.2)$, $L_c(.11)$ infinite diml.

(2) $L_c(\mathrm{triv}) = L(2.)$ fin diml $\Rightarrow 4c_1, 4c_2 \in \mathbb{Z}_{>0}$ or $2c_1 + 2c_2 \in \mathbb{Z}_{>0}$ or $4c_1, 2c_1 + 2c_2 \in \mathbb{Z}_{>0}$ or $4c_2, 2c_1 + 2c_2 \in \mathbb{Z}_{>0}$.

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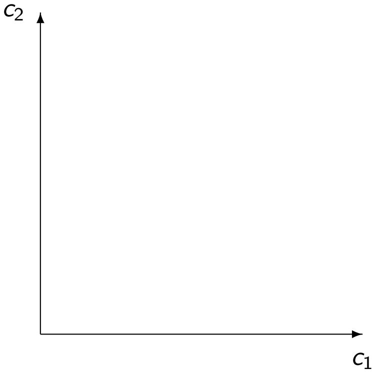
$B_2 = G(2,1,2)$

Making $c_1, c_2 \in \mathbb{Z}_{>0}$ varying, which modules can be finite dimensional?



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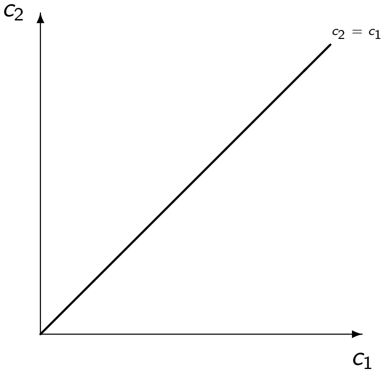

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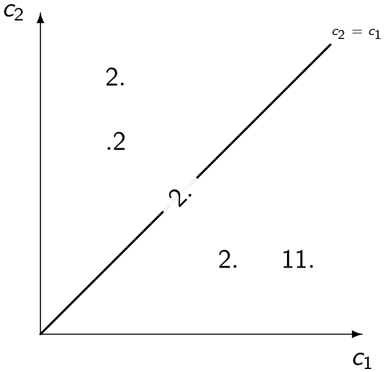

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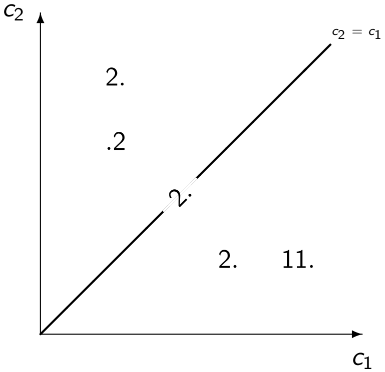

So $L_c(1.1)$ and $L_c(.2)$ are infinite dimensional for any choice of $c = (c_1, c_2)$.

The symmetric group-case: BEG classification

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Theorem (Berest-Etingof-Ginzburg)

The only possible finite dimensional irreducible module is $L_c(triv)$ and it is finite dimensional if and only if $c = r/n$, for $r \in \mathbb{N}$, $(r, n) = 1$.

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We want to see how far from the "truth" the condition given by our criterion is.

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$L_c(\lambda)$ cannot be finite dimensional.

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From our (weaker) necessary condition for Sym_n, it follows that

if $c > 0$ and $L_c(\lambda)$ is finite dimensional, then $\lambda = \mathrm{triv}$ and $nc \in \mathbb{Z}_{>0}$

E_6, E_7 and E_8

Applying our criterion we see that the only irreducible modules that can be finite dimensional for an appropriate $c > 0$ are (the ones corresponding to the following characters in GAP's notation):

- **$\mathbf{E_6}$**: $\phi(1,0) = \mathrm{triv}$, $\phi(6,1) = \mathrm{refl}$, $\phi(15,5)$ [3 out of 25]
- **$\mathbf{E_7}$**: $\phi(1,0) = \mathrm{triv}$, $\phi(7,1) = \mathrm{refl}$, $\phi(15,7)$, $\phi(21,6)$, $\phi(27,2)$, $\phi(35,13)$, $\phi(189,5)$ [7 out of 60]
- **$\mathbf{E_8}$**: $\phi(1,0) = \mathrm{triv}$, $\phi(8,1) = \mathrm{refl}$, $\phi(28,8)$, $\phi(35,2)$, $\phi(50,8)$, $\phi(56,19)$, $\phi(160,7)$, $\phi(175,12)$, $\phi(210,4)$, $\phi(300,8)$, $\phi(350,14)$, $\phi(560,5)$, $\phi(840,13)$, $\phi(840,14)$, $\phi(1050,10)$, $\phi(1400,8)$ [16 out of 122]

The "appropriate" values for the parameter c can also be easily found.

For example, for E_8, the conditions on c are as follows

(1) $L_c(\phi(1,0)) = L_c(\mathrm{triv})$ fin diml $\Rightarrow 30c \in \mathbb{Z}_{>0}$ or $24c \in \mathbb{Z}_{>0}$ or $20c \in \mathbb{Z}_{>0}$.

(2) $L_c(\phi(8,1)) = L_c(\mathrm{ref})$ fin diml $\Rightarrow 30c \in \mathbb{Z}_{>0}$ or $18c \in 2\mathbb{Z}_{>0} + 1$.

(3) $L_c(\phi(28,8))$ fin diml $\Rightarrow 20c \in 2\mathbb{Z}_{>0}$ or $18c \in 2\mathbb{Z}_{>0} + 1$ or $12c \in \mathbb{Z}_{>0}$.

(4) $L_c(\phi(35,2))$ or $L_c(\phi(50,8))$ fin diml $\Rightarrow 12c \in \mathbb{Z}_{>0}$.

(5) $L_c(\phi(56,19))$ fin diml $\Rightarrow 12c \in \mathbb{Z}_{>0}$.

(6) $L_c(\phi(160,7))$ fin diml $\Rightarrow 8c \in \mathbb{Z}_{>0}$ or $6c \in \mathbb{Z}_{>0}$.

(7) $L_c(\phi(175,12))$ or $L_c(\phi(300,8))$ fin diml $\Rightarrow 6c \in \mathbb{Z}_{>0}$.

(8) $L_c(\phi(210,4))$ fin diml $\Rightarrow 4c \in \mathbb{Z}_{>0}$ $6c \in 2\mathbb{Z}_{>0} + 1$.

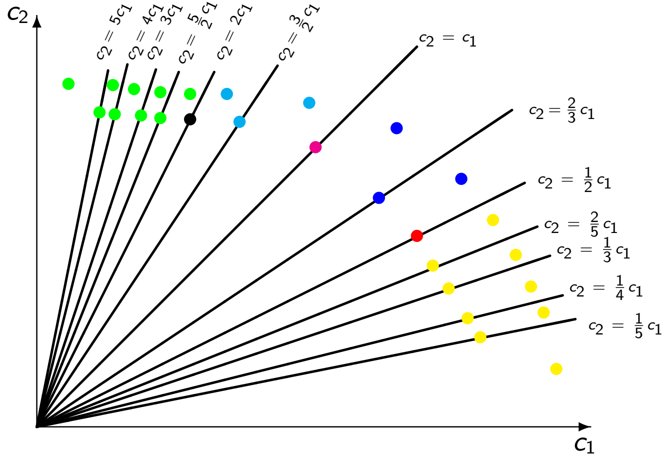
(9) $L_c(\phi(350,14))$ fin diml $\Rightarrow 4c \in 2\mathbb{Z}_{>0} + 1$.

(10) $L_c(\phi(840,13))$ fin diml $\Rightarrow 3c \in \mathbb{Z}_{>0}$.

(11) $L_c(\phi(560,5))$ or $L_c(\phi(840,14))$ or $L_c(\phi(1050,10))$ or $L_c(\phi(1400,8))$ fin diml $\Rightarrow 2c \in 2\mathbb{Z}_{>0} + 1$.

F_4

F_4 has 25 isoclasses of irreducibles and two conjugacy classes of reflections. If $c_1, c_2 > 0$, then the possible finite dimensional simples are



THANK YOU!

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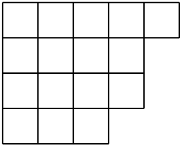

back

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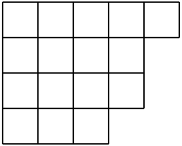

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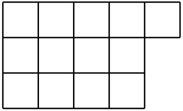
$\nu =$ 

$(k) =$ 

back

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0	1	2	$\cdots$	$n-1$

$c_\lambda = 0$

0	1	2	$\cdots$	$n-2$
-1				

$$c_\mu = c\left(\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} + 1\right) = cn$$

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