Quasi-hereditary algebras, A_∞-algebras and boxes

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24.03.2014

Quasi-hereditary algebras

k algebraically closed field

Definition

Let A be a finite dimensional k-algebra, $\{1, \ldots, n\}$ be an indexing set for the simple modules. A (w.r.t. to the linear order on $\{1, \ldots, n\}$) is called **quasi-hereditary** if there exist $\Delta(1), \ldots, \Delta(n) \in \operatorname{mod} A$ with

- $\operatorname{End}_A(\Delta(i)) = k$,
- $\operatorname{Hom}_A(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j$,
- $\operatorname{Ext}^1_A(\Delta(i), \Delta(j) \neq 0 \Rightarrow i < j$,
- ${}_A A \in \mathcal{F}(\Delta)$, where

$$\mathcal{F}(\Delta) := \{M | \exists 0 = M_0 \subset M_1 \subset \cdots \subset M_j = M \\ \forall i \exists k_i : M_i / M_{i+1} \cong \Delta(k_i)\}.$$

Examples of quasi-hereditary algebras and Δ's

Example

- $\mathfrak{g}$ complex semisimple, BGG-category $\mathcal{O} = \oplus \mathcal{O}_\chi = \oplus \text{mod}\, A_\chi$
  Δ Verma modules
- Schur algebra $S(n, r) = \text{End}_{k\mathfrak{S}_r}(V^{\otimes r}), V = k^n$
  Δ Weyl modules
- Algebras of global dimension ≤ 2
- $\text{Coh}(\mathbb{P}^n) \sim_{der} \text{mod}\, B_n$, the Beilinson algebra
  Δ exceptional collection

BGG category $\mathcal{O}$

Example

$\mathfrak{g}$ complex semisimple Lie algebra, $\mathfrak{b}$ Borel subalgebra
Verma modules $\Delta(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$

Theorem (Poincaré-Birkhoff-Witt)

$U(\mathfrak{g})$ is free over $U(\mathfrak{b})$. In particular $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} -$ is exact.

Question

Recall $\mathcal{O}_\chi \cong \operatorname{mod} A_\chi$. Does A_χ have a subalgebra with similar properties as $U(\mathfrak{b})$?

Exact Borel subalgebras

Definition (Koenig)

Let A be a quasi-hereditary algebra. $B \subseteq A$ is called an **exact Borel subalgebra** if the simples $L_B(i)$ of B can be indexed by $\{1, \ldots, n\}$ and

(E1) B is **directed**, i.e. quasi-hereditary with simple standard modules.

(E2) $A \otimes_B -$ is exact.

(E3) $\Delta_A(i) \cong A \otimes_B L_B(i)$.

Theorem (Koenig)

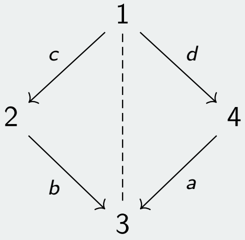
A_χ has an exact Borel subalgebra.

Existence?

Question

Does every quasi-hereditary algebra have an exact Borel subalgebra?

Example



$$\ell(\Delta(i)) = \begin{cases} 1 & \text{for } i \neq 4 \\ 2 & \text{for } i = 4 \end{cases}$$

$KQ/\langle ad - bc \rangle$ is quasi-hereditary without exact Borel subalgebra.

Existence!

Theorem (Koenig-K-Ovsienko)

For every quasi-hereditary algebra A there is $R \sim_{Mor} A$ such that R has an exact Borel subalgebra.

Example

$$\begin{pmatrix} * & * & * & * & * \\ & * & 0 & 0 & * \\ & & * & * & * \\ & & * & * & * \\ & & & & * \end{pmatrix} \supset \begin{pmatrix} * & \dagger & \dagger & * & 0 \\ & \diamond & 0 & 0 & -\bullet \\ & & \diamond & * & \bullet \\ & & & * & 0 \\ & & & & * \end{pmatrix}$$

is an exact Borel subalgebra of a Morita equivalent algebra.

Sketch of the Proof

A quasi-hereditary
$\mathcal{F}(\Delta_A)$

$\longmapsto$ Kadeishvili / Keller-Lefèvre

$\mathcal{E} = \mathrm{Ext}^*(\Delta, \Delta)$ A_∞-algebra
$H^0(\mathrm{twmod}\,\mathcal{E})$

$\downarrow$ KKO

$\mathfrak{B} = (B, W)$ box
$\mathrm{mod}\,\mathfrak{B}$

$\longmapsfrom$ Burt-Butler

$R = \mathrm{End}_{\mathfrak{B}}(B)$ quasi-hereditary
$\mathcal{F}(\Delta_R)$

$\updownarrow$ Dlab-Ringel

Outline

1. Quivers with relations
2. A-infinity algebras and reconstruction
3. Boxes = Corings

Gabriel's Theorem

Theorem

Any finite dimensional algebra A is Morita equivalent to the path algebra of a quiver Q with relations ρ, i.e.

$$\operatorname{mod} A \cong \operatorname{mod} KQ/(\rho).$$

Quivers with relations $KQ/(\rho)$

Definition

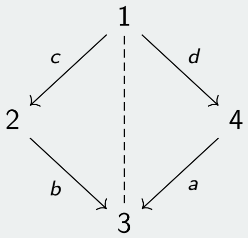
- A **quiver** is an oriented graph (cycles, double arrows, loops allowed).
- The **path algebra** KQ is

$$\{\sum_j \lambda_j p_j | p_j \text{ path in } Q \}$$

  including stationary paths e_i for each vertex i. These are primitive idempotents.
- **Multiplication** is concatenation of paths if possible, 0 otherwise.
- **Relations**: Generators of (ρ) can be chosen to be linear combinations of paths $i \to i'$ of length ≥ 2.

The commuting square

Example



$KQ/(ad - bc)$ has a basis $\{e_1, e_2, e_3, e_4, a, b, c, d, ad = bc\}$.

- Simple modules $\leftrightarrow$ primitive idempotents $\leftrightarrow$ vertices
- Projective modules $\leftrightarrow$ paths from a fixed vertex i
- Standard modules $\Delta \leftrightarrow$ paths from i to lower vertices

How to get (Q, ρ) for an algebra?

- vertices $\leftrightarrow$ isoclasses of simple modules $L(i)$,
- arrows $\#\{i \to j\} = \dim \mathrm{Ext}^1(L(i), L(j))$,
- relations $\#\{\ i \dashrightarrow j\ \} = \dim \mathrm{Ext}^2(L(i), L(j))$.

  Usually not sufficient, need extra data

$\rightsquigarrow$ A_∞-algebras

A-infinity algebras

Definition

An A_∞-**algebra** A is a graded vector space together with operations $m_n : A^{\otimes n} \to A$ of degree $2 - n$ satisfying:

- $m_1^2 = 0 : A \to A$
- $m_2(m_1 \otimes 1 + 1 \otimes m_1) = m_1 m_2 : A^{\otimes 2} \to A$
- $m_2(m_2 \otimes 1 - 1 \otimes m_2) =$
  $m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1) : A^{\otimes 3} \to A$
- ...

The Yoneda A_∞-algebra

M a module, then $\text{Ext}^*(M, M)$ is an A_∞-algebra with

- $m_1 = 0$
- m_2 the Yoneda product (i.e. concatenation of exact sequences)
- $m_{\geq 3}$ given inductively by choosing representatives of Ext in $\text{Hom}^*(P^\bullet, P^\bullet)$,
  are measuring the difference between homotopic and equal

Example

Known for:

- $\text{Ext}^*_{C_p}(k, k)$ [Madsen], $\text{Ext}^*_{S_p}(k, k)$ [Schmid], $p = \text{char}(k)$
- partially for $\text{Ext}^*_G(k, k)$ with $G \in \{C_l \times C_m, D_8, D_{16}, Q_8\}$ [Vejdemo-Johansson]
- $\text{Ext}^*_{\mathcal{O}^{\mathfrak{p}}}(\Delta, \Delta)$ for $\mathfrak{gl}$ in small cases [Klamt-Stroppel]

Relations via A_∞-structure

Theorem (Keller-Lefèvre)

Let $S = \oplus L(i)$. Then the A_∞-structure on

$$\mathrm{Ext}^1(S,S) \oplus \mathrm{Ext}^2(S,S)$$

determines (Q,ρ).

Idea of Proof.

$$m_n : \mathrm{Ext}^1 \otimes \cdots \otimes \mathrm{Ext}^1 \to \mathrm{Ext}^2$$

Dualise:

$$D\,\mathrm{Ext}^2 \to \bigoplus_n (D\,\mathrm{Ext}^1)^{\otimes n}$$

images of a basis of Ext^2 correspond to relations. □

Reconstruction of $\mathcal{F}(\Delta)$

Theorem (Keller-Lefèvre)

Let $\Delta = \oplus \Delta(i)$. $\mathcal{F}(\Delta)$ is determined by the A_∞-structure on

$$\mathrm{Hom}(\Delta, \Delta) \oplus \mathrm{Ext}^1(\Delta, \Delta) \oplus \mathrm{Ext}^2(\Delta, \Delta)$$

Idea of proof.

Realise $\mathcal{F}(\Delta)$ as the following category:
an object is:

- $(\Delta(i_1), \ldots, \Delta(i_t))$
- $\delta_{jk} \in \mathrm{Ext}^1(\Delta(i_j), \Delta(i_k))$
- compatibility from m_n

a morphism is:

- morphisms $f_{jk} \in \mathrm{Hom}(\Delta(i_j), \Delta(i_k))$ + compatibility from m_n $\square$

Directed Boxes

Theorem (Koenig - K - Ovsienko)

The A_∞-structure on $\mathrm{Hom}(\Delta, \Delta) \oplus \mathrm{Ext}^1(\Delta, \Delta) \oplus \mathrm{Ext}^2(\Delta, \Delta)$ *defines a directed box* $\mathfrak{B} = (B, W)$*, such that*

$$\mathrm{mod}\, \mathfrak{B} \cong \mathcal{F}(\Delta)$$

The algebra B is an exact Borel subalgebra of $R = \mathrm{End}_{\mathfrak{B}}(B)$*.*

Idea of Proof.

Dualise the m_n. □

Boxes

Definition

A **box** $\mathfrak{B} = (B, W) = (B, W, \mu, \varepsilon)$ is:

- an algebra B,
- a B-**coring** W, i.e.
  - a B-B-bimodule W,
  - a coassociative comultiplication $\mu : W \to W \otimes_B W$,
  - a counit $\varepsilon : W \to B$.

Definition

The category mod $\mathfrak{B}$ is the full subcategory of the category of W-comodules of induced comodules $W \otimes_B M$ for a B-module M. Comodule structure on $W \otimes_B M$:

$$(W \otimes_B M) \overset{\mu \otimes 1}{\to} W \otimes_B (W \otimes_B M).$$