Non-crossing partitions and hereditary algebras

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Hereditary algebras

Let A be an hereditary artin algebra.

This means that every submodule of a projective module is again hereditary.

Equivalently, $\mathrm{Ext}^2_A(-,-)$ vanishes.

Example

The path algebra $A = KQ$ of a quiver Q without oriented cycles.

Exceptional sequences

Call $X = (X_1, \dots, X_r)$ an **exceptional sequence** in $\operatorname{mod} A$ if

- each X_i is an indecomposable A-module.
- $\operatorname{Hom}_A(X_i, X_j) = 0$ for $i > j$.
- $\operatorname{Ext}^1_A(X_i, X_j) = 0$ for $i \geq j$.

Set $C(X) \subset \operatorname{mod} A$ to be smallest subcategory containing X and closed under kernels, cokernels and extensions. Call X **full** if $C(X) = \operatorname{mod} A$.

Example

We can order the simple A-modules to form a full exceptional sequence.

Properties

Let $X = (X_1, \ldots, X_r)$ be an exceptional sequence.

Lemma (Happel–Ringel)

Each $\mathrm{End}_A(X_i)$ *is a division algebra.*

Theorem (Geigle–Lenzing)

$C(X)$ *is equivalent to* $\mathrm{mod}\, B$ *for some hereditary artin algebra* B.

We therefore have a full embedding $\mathrm{mod}\, B \to \mathrm{mod}\, A$.

Aim: describe all such subcategories.

Homological epimorphisms, I

A ring homomorphism $A \to B$ is a **finite homological epimorphism** if

- B is finitely generated as an A-module.
- restriction of scalars induces isomorphisms

$$\mathrm{Ext}^n_B(X, Y) \cong \mathrm{Ext}^n_A(X, Y)$$

for all $n \geq 0$ and all $X, Y \in \mathrm{mod}\, B$.

Homological epimorphisms, II

Let A be hereditary and X an exceptional sequence.

Then the inclusion

$$C(X) \to \operatorname{mod} A$$

admits both left and right adjoints.

So, if $C(X) \cong \operatorname{mod} B$, then we obtain a finite homological epimorphism $A \to B$.

Conversely, if $A \to B$ is a finite homological epimorphism, then $\operatorname{mod} B \cong C(X)$ for some exceptional sequence X.

A category of hereditary categories

We can rephrase our aim as describing the category $\mathfrak{H}$, having

objects	mod A for some hereditary artin algebra A.
morphisms	mod $B \to$ mod A induced from some finite homological epimorphism $A \to B$.

Mutations

Let $X = (X_1, \dots, X_r)$ be an exceptional sequence.

Lemma (Crawley-Boevey,Ringel)

Assume X is not full. For each i there is an exceptional sequence

$$(X_1, \dots, X_{i-1}, Y, X_i, \dots, X_r).$$

Thus for each i we have the **mutations**

$$\sigma_i(X) := (X_1, \dots, X_{i-1}, X_{i+1}, X', X_{i+2}, \dots, X_r).$$
$$\sigma_i^{-1}(X) := (X_1, \dots, X_{i-1}, X'', X_i, X_{i+2}, \dots, X_r).$$

and which leave the subcategory $C(X)$ unchanged.

Braid group action

Theorem (Crawley-Boevey,Ringel)

Mutations yield an action of the braid group on exceptional sequences.
Moreover, two exceptional sequences lie in the same orbit if and only if they determine the same subcategory.

Generalised Cartan lattices, I

A **bilinear lattice** is a pair $(\Gamma, \langle -, - \rangle)$ where

- Γ is a finite free abelian group.
- $\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}$ is a non-degenerate bilinear form.

An element $x \in \Gamma$ is a **(pseudo) real root** if

- $\langle x, x \rangle > 0$.
- $\dfrac{\langle a, x \rangle}{\langle x, x \rangle}$ and $\dfrac{\langle x, a \rangle}{\langle x, x \rangle}$ are integers, for all $a \in \Gamma$.

Generalised Cartan lattices, II

A **generalised Cartan lattice** is a bilinear lattice together with a basis $E = (e_1, \ldots, e_n)$ such that

- each e_i is a real root.
- $\langle e_i, e_j \rangle = 0$ for $i > j$.
- $\langle e_i, e_j \rangle \leq 0$ for $i < j$.

Grothendieck groups

The generalised Cartan lattices are precisely the Grothendieck groups of hereditary artin algebras A.

- $\Gamma = K_0(A)$
- $\langle [X], [Y] \rangle = \dim \operatorname{Hom}_A(X, Y) - \dim \operatorname{Ext}^1_A(X, Y)$.
- E is the basis given by the simple modules, ordered to form a full exceptional sequence.

Weyl groups

Let $(\Gamma, \langle -,- \rangle, E)$ be a generalised Cartan lattice.

For $e \in E$ define the **simple reflection**

$$s_e \colon x \mapsto x - \frac{\langle x, e\rangle + \langle e, x\rangle}{\langle e, e\rangle} e.$$

The **Weyl group** W is the group of automorphisms of Γ generated by the simple reflections.

Roots and Reflections

The **real roots** are

$$\Phi := \{w(e) : w \in W,\, e \in E\}.$$

The **reflections** in W are

$$T := \{s_x : x \in \Phi\} = \{ws_e w^{-1} : w \in W,\, e \in E\}.$$

Absolute length

Let W be a Weyl group, with reflections T.

The **absolute length** $\ell(w)$ of $w \in W$ is the minimal r such that

$$w = t_1 t_2 \dots t_r, \quad t_i \in T.$$

The **absolute order** is given by

$$w \leq v \quad \text{provided} \quad \ell(v) = \ell(w) + \ell(w^{-1}v).$$

Non-crossing partitions

The **Coxeter element** is

$$c := s_{e_1} s_{e_2} \cdots s_{e_n}.$$

This has absolute length n.

The poset of **non-crossing partitions** is

$$NC(W, c) := \{w \in W : w \leq c\}.$$

A category of generalised Cartan lattices

A morphism of generalised Cartan lattices

$$\phi\colon (\Gamma', \langle -,-\rangle', E') \to (\Gamma, \langle -,-\rangle, E)$$

consist of an isometry $\phi\colon \Gamma' \to \Gamma$ such that, if $E' = (e'_1, \ldots, e'_n)$, then

- each $\phi(e'_i)$ is a real root of Γ.
- $s_{\phi(E')} = s_{\phi(e'_1)} s_{\phi(e'_2)} \cdots s_{\phi(e'_n)}$ is a non-crossing partition for Γ.

This gives a category $\mathfrak{C}$ of generalised Cartan lattices.

Theorem

Let A be an hereditary artin algebra.

If $X = (X_1, \ldots, X_r)$ is an exceptional sequence, then each $[X_i] \in K_0(A)$ is a real root.

The map $X \mapsto s_X := s_{X_1} s_{X_2} \cdots s_{X_r}$ yields a bijection between

- orbits of exceptional sequences under the braid group action,

  equivalently the subcategories $C(X)$ of mod A.
- non-crossing partitions in $K_0(A)$.

This is an isomorphism of posets, so

$$C(X) \subseteq C(Y) \quad \Leftrightarrow \quad s_X \leq s_Y$$

Sketch of proof, I

We have that

- every exceptional sequence can be completed to a full exceptional sequence.
- full exceptional sequences form a single orbit under the braid group.
- if X is full, then $s_X = c$.

It follows that

- if X is an exceptional sequence, then s_X is a non-crossing partition.
- if $C(X) \subseteq C(Y)$, then $s_X \leq s_Y$.

Hence we have a map of posets from subcategories of the form $C(X)$ to $NC(W, c)$.

Sketch of proof, II

Surjectivity follows from:

Theorem (Igusa–Schiffler)

There is a transitive braid group action on sequences of reflections $(t_1, t_2, \ldots, t_n)$ such that $t_1 t_2 \cdots t_n = c$.

[New proof: Baumeister–Dyer–Stump–Wegener.]

Injectivity: given X, there is an exceptional sequence Z such that $s_X s_Z = c$. Then $C(X)$ coincides with the **right perpendicular category** $Z^\perp$. So if $s_X = s_Y$, then $C(X) = Z^\perp = C(Y)$.

Note: injectivity is the only part we cannot prove combinatorially.

A functorial interpretation, I

There is a functor

$$F\colon \mathfrak{H} \to \mathfrak{C}, \quad \operatorname{mod} A \mapsto K_0(A).$$

On morphisms, if $f\colon A \to B$ is a finite homological epimorphism, then restriction of scalars gives the morphism

$$f^*\colon \operatorname{mod} B \to \operatorname{mod} A$$

and this induces a morphism of generalised Cartan lattices

$$F(f^*)\colon K_0(B) \to K_0(A).$$

A functorial interpretation, II

- If $f^*, g^*\colon \operatorname{mod} B \to \operatorname{mod} A$ are two morphisms in $\mathfrak{H}$, then

$$F(f^*) = F(g^*) \quad \Leftrightarrow \quad f^* \cong g^* \quad \text{(nat. iso.)}$$

- The functor F is not full. Example: if K and L are fields, then $K_0(K) = K_0(L)$ but $\operatorname{mod} K \cong \operatorname{mod} L$ implies $K \cong L$.
- Every morphism of generalised Cartan lattices is, up to isomorphism, of the form $F(f^*)$ for some f^* in $\mathfrak{H}$.

Functoriality of Weyl groups, I

Recall that a morphism of generalised Cartan lattices

$$\phi\colon (\Gamma', \langle -, - \rangle', E') \to (\Gamma, \langle -, - \rangle, E)$$

satisfies that $s_{\phi(E')} \in NC(W, c)$.

It follows that the poset of subobjects of Γ is isomorphic to the poset of non-crossing partitions.

Functoriality of Weyl groups, II

In fact, more is true.

The morphism ϕ induces an injective group homomorphism

$$\phi_*\colon W' \to W, \quad s_{e'} \mapsto s_{\phi(e')}.$$

This restricts to an injective map of posets

$$NC(W', c') \to NC(W, c).$$

Sketch of proof

Let $\overline{W} \leq W$ be the subgroup generated by the reflections $s_{\phi(e')}$ for $e' \in E'$. Then restriction to the sublattice $\phi(\Gamma')$ yields a surjective group homomorphism $\overline{W} \to W'$.

We need to prove that this is injective, so that any relation satisfied by the $s_{\phi(e')}$ is also satisfied by the $s_{e'}$.

By induction we can reduce to when $\phi(\Gamma')$ has corank one, and then to the case when W is affine.

Now use the explicit description of W as a semi-direct product of a finite Weyl group by the root lattice.

Summary

We now have a category $\mathfrak{C}$ of generalised Cartan lattices, which captures many of the salient features of the category $\mathfrak{H}$ of categories $\operatorname{mod} A$ for hereditary artin algebras, via the functor

$$\mathfrak{H} \to \mathfrak{C}, \quad \operatorname{mod} A \mapsto K_0(A).$$

Moreover, we have functors

$$\mathfrak{C} \to \textbf{Groups}, \qquad \mathfrak{C} \to \textbf{Posets}$$

sending Γ to its Weyl group W and its poset of non-crossing partitions $NC(W, c)$.

In particular, the poset of subcategories $C(X) \subset \operatorname{mod} A$ is isomorphic to non-crossing partitions for $K_0(A)$.

Pointed Weyl groups, I

One can now ask:

What subgroups of the Weyl group arise in this way?

Note: each Coxeter element $c \in W$ determines (an ordering of) a simple system, up the the action of the braid group.

We are therefore interested, not in Coxeter systems (W, S), but in pairs (W, c), which we call **pointed Weyl groups**.

Pointed Weyl groups, II

A monomorphism of pointed Weyl groups is an injective group homomorphism $\theta\colon W' \to W$ such that

- $\theta(c') \in NC(W, c)$.
- $\ell(\theta(c')) = \ell(c')$.

It follows that

$$NC(W', c') \cong \{w \in NC(W, c) : w \leq \theta(c')\}.$$

Pointed Weyl groups, III

If $\phi\colon \Gamma' \to \Gamma$ is a morphism of generalised Cartan lattices, then it induces a monomorphism of pointed Weyl groups $\phi\colon (W', c') \to (W, c)$.

Moreover, we obtain each such monomorphism of pointed Weyl groups in this way.

Pointed Weyl groups, IV

Example

If W is finite, then every such subgroup is a parabolic subgroup.

Example

Let $W = \langle s, t, u\rangle$ be affine of type $\widetilde{A}_2$, with Coxeter element $c = stu$, so

$$(st)^3 = (su)^3 = (tu)^3 = 1.$$

Then $c = s(utu)(ututu)$ and $W' = \langle s, utu\rangle$ is affine of type $\widetilde{A}_1$. Thus subgroups of pointed Weyl groups are in general not parabolic subgroups.

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