On the number of tilting modules via polytopes and the Markov equation (without Riemann–Roch)

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Introduction

Q a Dynkin quiver (n vertices): $\mathbb{A}_n$, $\mathbb{B}_n$, $\mathbb{C}_n$, $\mathbb{D}_n$, $\mathbb{E}_{6,7,8}$, $\mathbb{F}_4$, or $\mathbb{G}_2$
$\Phi = \Phi(Q)$ the corresponding root system, Φ^+ the positive roots, and Π the simple roots.

There are two different points of view:
– we consider Φ inside the root lattice $R(\Phi)$ with its euclidean metric
– we consider the standard basis Π and coordinates given by a decomposition $\alpha = \sum_{i=1}^n d_i \alpha_i$ (dimension vector)

A module M over $A = kQ$ is *rigid* if $\mathrm{Ext}^1(M, M) = 0$. Thus M is dense in the space of all representations.

principal aim: count the number of rigid modules $r^+(Q)$ and the number of tilting modules $t^+(Q)$ (up to isomorphism and multiple direct summands).
A basic rigid module (no multiple direct summands) has at most n direct summands. We define $r^{(m)}(Q)$ to be the number of rigid modules with m pairwise non-isomorphic direct summands.

Quivers of type $\mathbb{A}$

We define several polytopes associated to the root system Φ:

$$C(\Phi) = \text{conv}\{\Phi\} = P(Q);\, C^+(\Phi) = \text{conv}\{\Phi^+, 0\} = P^+(Q);$$
$$C^{\text{clus}}(\Phi) = \text{conv}\{\Phi^+, -\Pi\} = P^{\text{clus}}(Q)$$

Theorem

We have
$\text{vol}\, P^+(Q) = t^+(Q);\ \text{vol}\, P^{\text{clus}}(Q) = t^{\text{clus}}(Q);\ \text{vol}\, P(Q) = t(Q).$

remark for the other types:
– the polytopes $P^*(Q)$ are always *reflexive* (the lattice distance of any facet to zero is one).
– the polytopes $C^*(\Phi)$ are always convex, however in general NOT reflexive (thus $P(Q) \neq C(\Phi)$).
– on the polytopes $C^*(\Phi)$ we have an action of the Weyl group and an action of the Coxeter transformation ($\mathbb{Z}/h$ acts)

Quivers of type $\mathbb{A}$ and recursion formulas

remark:
– If P is reflexive, we get $\operatorname{vol} P = \sum \operatorname{vol} F$, where F runs through the set of facets.
– If P^+ is half reflexive (zero is a vertex and all facets not containing zero are reflexive) we get $\operatorname{vol} P = \sum_{0 \notin F} \operatorname{vol} F$.

We compute the volume of the polytopes $P(\mathbb{A}_n)$ using the following formulas:

$\operatorname{vol} P(Q) = \operatorname{vol} C(Q) = \sum_{i=1}^{n} |W / \operatorname{Stab}(i)| \operatorname{vol} F_i$
$= |W| \operatorname{vol} F = h |\operatorname{vol} G|$,
where F is the fundamental domain w.r.t. W and G the fundamental domain w.r.t. the Coxeter transformation

$\operatorname{vol} P(Q) = \sum \operatorname{vol} P^+(Q(I)) \operatorname{vol} P^+(Q \setminus I)$
$\operatorname{vol} P^{\mathrm{clus}}(Q) = \sum \operatorname{vol} P^+(Q(I))$
$\operatorname{vol} P^+(Q) = \sum \operatorname{vol} P^+(Q \setminus \{i\})$

The definition of the polytopes

We define $C(Q)$ to be the convex hull of the roots and $P(Q)$ the union of simplices formed by some subsets of roots.

Let T be any rigid module (it is dense in the space of all representations). If T is indecomposable, $\underline{\dim}\, T$ is a positive root. If T is not indecomposable $T = \oplus T(i)^{a_i}$. This defines a simplicial complex $P^+(Q)$, $P^{\mathrm{clus}}(Q)$, and $P(Q)$, that we make into a polytope: the vertices are just the roots.

$$P^+(Q) = \bigcup_{T \in \mathcal{T}(Q)} \overline{\sigma}_T$$

Problem: Is the definition correct? If YES, everything follows immediately (with TLP).

The reflexive polytopes

Theorem

The polytopes $P^+(Q)$ are reflexive. In particular, the volume is the sum of the volumes of the facets (the positive facets). Thus $P^+(Q) = t^+(Q)$.

Take a tilting module T and the corresponding positive roots $\{\underline{\dim}\, T(i) \mid i = 1, \ldots, n\}$. They form a $\mathbb{Z}$–basis of the root lattice. Thus the hyperplane $H(T) = \{x \in R_{\mathbb{R}} \mid \phi(x) = 1\}$ for $\phi(\dim T(i)) = 1$ is reflexive.

Theorem

The polytopes $C(Q)$ (and $C^+(Q), C^{\mathrm{clus}}(Q)$) are reflexive for type $\mathbb{A}$, $\mathbb{C}$ ($\mathbb{B}_2$), $\mathbb{D}$ and $\mathbb{E}_6$.

Proof (idea). TTT: make a table.
Each facet is of the form $e_i^+ = a$ for some basis vector e_i up to automorphism. The maximal possible value for a is d_i, where d is the maximal positive root.

Convex polytopes and concave pieces of some polytopes

Theorem

The following properties are equivalent:
The polytope $P(Q)$ ($P^{+}(Q)$, $P^{\mathrm{clus}}(Q)$) is convex.
The quiver Q is of type $\mathbb{A}$ or $\mathbb{C}$ (including $\mathbb{B}_2$).
In the Auslander-Reiten quiver of $kQ-\mathrm{mod}$ each AR-sequence has at most three middle terms.

We have three minimal concave pieces:
$\mathbb{G}_2$: $M \longrightarrow N^3 \longrightarrow M'$
$\mathbb{B}_3$: $M \longrightarrow N_1 \oplus N^2 \longrightarrow M'$
$\mathbb{D}_4$: $M \longrightarrow N_1 \oplus N_2 \oplus N_3 \longrightarrow M'$

Theorem

We compare the volume of $C(Q)$ with the volume of $P(Q)$ for the minimal concave cases:
$\mathrm{vol}\, P(Q) = \mathrm{vol}\, C(Q) - h$

The Markov equation

Theorem (Beineke, Brüstle, H.)

The following equation is invariant under $(x, y, z) \mapsto (x, y, xy - z)$ *and it is invariant under the action of* S_3*:*

$$x^2 + y^2 + z^2 - xyz = \mathrm{Mark}(x, y, z).$$

It is a derived invariant for any algebra with three vertices: the Markov constant of the upper triangular matrix $(\hom(T(i), T(j)))_{i,j=1}^3$ *depends only on* Q *(and not on the orientation of* Q *or the choice of* T*).*

example: $X = \mathbb{P}^2$ then $\mathrm{Mark}(\mathbb{P}^2) = 0$
$Q = \mathbb{A}_3$ then $\mathrm{Mark}(\mathbb{A}) = 2$
$Q = \mathbb{B}_3$ "=" $\mathbb{C}_3$ then $\mathrm{Mark}(\mathbb{B}_3) = 3$, but $x^2 + y^2 + z^2 - xyz \neq 3$ for any $x, y, z \in \mathbb{Z}$!!!

Need to define the Markov equation for species and for any n.

The general Markov equation (Conjecture)

Theorem (Bondal, H.)

The following equations F_i are invariant under $\sigma_i : (x_{i,j}) \mapsto (y_{i,j})$ and the action of $x_{i,j} \mapsto x_{i+1,j+1}$: F_i has lowest terms $\sum G(2i)^2$, where $G(2i)$ is the $2i$-th Pfaffian and the sum is taken over all $2i$ element subsets of $\{1, \ldots, n\}$. It is a derived invariant for any algebra (smooth projective algebraic variety) with n vertices: the Markov constant of the upper triangular matrix $(\hom(T(i), T(j)))_{i,j=1}^n$ depends only on Q (and not on the orientation of Q or the choice of T). The polynomials are already defined over the species.

The polynomials F_i are linear combinations of the coefficients of the Coxeter polynomial. Consequently, it is a derived invariant.

Open conjecture: these are all invariants (each invariant is a polynomial in the F_i or G): $\mathbb{Z}[x_{i,j}]^{B_n} = \mathbb{Z}[x_{i,j}]^H$, where H is an algebraic group (the orthogonal group with respect to the Euler form).

Recursion formulas and Proofs

Theorem

Let Q be a Dynkin quiver (or a tame quiver), then

$$\sum_{T \in \mathcal{T}(Q)} \prod_{i}^{n} 1/\dim T(i)$$

Theorem

The sets $P^{+}(Q)$, P^{clus}, and $P(Q)$ are polytopes (no gaps and no overlaps).

Theorem

$$\operatorname{vol} P^{+}(Q) = \sum \operatorname{vol} P^{+}(Q \setminus \{i\}) + \operatorname{cor}(\Phi),$$

where the correction term can be computed for $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ using $\mathbb{Z}/2$–actions.

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Wild Quivers

Let Q be wild, and assume it is very wild (many ≥ 2 arrows between any two vertices). Then we define a certain convex set, it is an intersection of a countable number of polytopes

$$P = \bigcap_{i \in \mathbb{Z}} \Phi^i \Delta.$$

Moreover, $v^+ = \lim_{i \to \infty} \Phi^i d / \rho^i$ and $v^+ = \lim_{i \to -\infty} \Phi^i d / \rho^i$ are in P.

Theorem (De la Pena, H., Kerner)

The set P is polyhedral, except in the two limit points v^+ and v^-, and each facet is of the form $P(\mathbb{A}_{n-2}) = C(\mathbb{A}_{n-2})$.

THANK YOU FOR YOUR ATTENTION

and I missed my black boards