Orlov's Equivalence and Maximal Cohen-Macaulay Modules over the Cone of an Elliptic Curve

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Outline

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Let $(R, \mathfrak{m})$ be a noetherian local ring and M a finitely generated R-module. M is said to be a maximal Cohen-Macaulay (MCM) R-module if $\mathrm{Ext}^i_R(K, M) = 0$ for $i < \dim(R)$, where $K = R/\mathfrak{m}$ denotes the residue field.

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Example

1. $\dim(R) = 1$, R reduced: $M \in \mathrm{MCM}(R) \iff M$ is torsionfree.
2. $\dim(R) = 2$, R normal: $M \in \mathrm{MCM}(R) \iff M$ is reflexive.

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2. $$S^n \xrightarrow{A} S^n \xrightarrow{\operatorname{adj}(A)} S^n$$
is a matrix factorisation of $\det(A)$.

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$(A, B) \mapsto \mathrm{cok}(B)$ *induces an exact equivalence of triangulated categories*

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Theorem (Buchweitz)

The inclusion $\mathrm{MCM}(R) \subseteq R - \mathrm{mod}$ *induces an exact equivalence of triangulated categories*

$$\underline{\mathrm{MCM}}(R) \cong \mathscr{D}^b(R - \mathrm{mod})/\mathscr{P}(R),$$

where $\mathscr{P}(R)$ *denotes the finite complexes of finitely generated free* R*-modules.*

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Theorem (Kahn)

1. *The indecomposable* MCM *R-modules can be enumerated (up to isomorphism) by $M_{r,d}(\lambda)$ with $\lambda \in E^{\text{closed}}$, $(r, d) \in \mathbb{Z}^2$ such that $r \geq 1$, $r \leq d < 4r$.*

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2. *The Auslander-Reiten quiver of* $\mathrm{MCM}(R)$ *is given as a disjoint union of tubes*

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Question

Concrete description of all indecomposables?

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Goal

Generalise this to arbitrary τ.

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Let $A = K[X_0, \ldots, X_n]/(f)$, f homogeneous, $X = \mathrm{Proj}(A)$. Denote by $\mathscr{T}(A)$ the triangulated subcategory of $\mathscr{D}^b(grA)$ generated by all finite dimensional graded A-modules and $\mathscr{P}(A)$ the triangulated subcategory of $\mathscr{D}^b(grA)$ generated by all finitely generated free modules.

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$$\Big\| \; \textit{Serre}$$

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$$\mathscr{D}^b(grA) \longrightarrow \mathscr{D}^b(grA)/\mathscr{P}(A)$$

$$\mathscr{D}^b(grA)/\mathscr{P}(A) \overset{Buchweitz, Orlov}{=\!=} \underline{MF}^{gr}(f)$$

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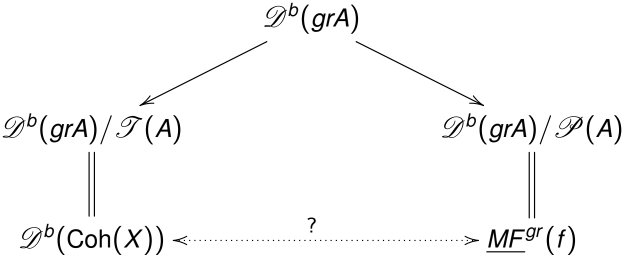

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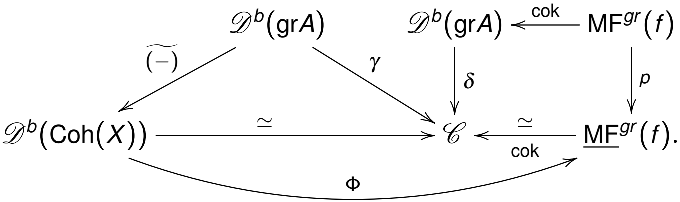

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$$\begin{array}{ccccccc} & & \mathscr{D}^b(\mathrm{gr}A) & & \mathscr{D}^b(\mathrm{gr}A) & \xleftarrow{\mathrm{cok}} & \mathrm{MF}^{gr}(f) \\ & \swarrow^{\widetilde{(-)}} & & \searrow^{\gamma} & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle p} \\ \mathscr{D}^b(\mathrm{Coh}(X)) & & \xrightarrow{\simeq} & & \mathscr{C} & \xleftarrow[\mathrm{cok}]{\simeq} & \underline{\mathrm{MF}}^{gr}(f). \end{array}$$

$$\mathscr{D}^b(\mathrm{Coh}(X)) \xrightarrow{\Phi} \underline{\mathrm{MF}}^{gr}(f)$$

γ is given as the composite

$$\mathscr{D}^b(\mathrm{gr}A) \xrightarrow{\mathrm{tr}_{\geq 1}} \mathscr{D}^b(\mathrm{gr}A) \xrightarrow{\mathrm{RHom}(-,A)} \mathscr{D}^b(\mathrm{gr}A) \xrightarrow{\mathrm{tr}_{\geq 0}} \mathscr{D}^b(\mathrm{gr}A) \xrightarrow{\mathrm{RHom}(-,A)} \mathscr{D}^b(\mathrm{gr}A).$$

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To compute $\Phi(\mathscr{O}_X)$, one should

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Example Computation

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Let E be the elliptic curve associated to $Y^2 = X^3 + aX + b$. Then

$$\Phi(\mathscr{O}_E) = R(-3) \oplus R(-4)^3 \xrightarrow{\begin{pmatrix} Z & YZ & X^2 & 0 \\ -Y & -bZ^2 & aYZ & X^2 + aZ^2 \\ X & 0 & -bZ^2 - aXZ & -YZ \\ 0 & X & Y & Z \end{pmatrix}}$$

$$R(-2)^3 \oplus R(-3) \xrightarrow{\begin{pmatrix} -bZ^2 - aXZ & -YZ & -X^2 & aZ^2Y \\ Y & Z & 0 & -X^2 - aZ^2 \\ -X & 0 & Z & YZ \\ 0 & -X & -Y & -bZ^2 \end{pmatrix}} R \oplus R(-1)^3.$$

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$$\ldots \to A(-5)^3 \oplus A(-6) \xrightarrow{\begin{pmatrix} -bZ^2 - aXZ & -YZ & -X^2 & aZ^2Y \\ Y & Z & 0 & -X^2 - aZ^2 \\ -X & 0 & Z & YZ \\ 0 & -X & -Y & -bZ^2 \end{pmatrix}}$$

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Finishing the Calculation

Now we know the matrix factorisation looks something like

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We just need to apply the correct shift functor (in our case: once).

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Example

$$S^3 \xrightarrow{\begin{pmatrix} -X^2-\lambda XZ-(a+\lambda^2)Z^2 & -Z(Y+\mu Z) & \lambda\mu Z^2+XY+\mu XZ+\lambda YZ \\ -X(Y-\mu Z) & -X(X-\lambda Z) & -(a+\lambda^2)XZ+Y^2-bZ^2 \\ -Z(Y-\mu Z) & -Z(X-\lambda Z) & X^2+\lambda^2 Z^2 \end{pmatrix}}$$

$$S^3 \xrightarrow{\begin{pmatrix} X-\lambda Z & 0 & -Y-\mu Z \\ \mu Z-Y & X+\lambda Z & (a+\lambda^2)Z \\ 0 & Z & -X \end{pmatrix}} S^3$$

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\mathscr{D}^b(\mathrm{Coh}(E)) & \xrightarrow{\quad\Phi\quad} & \underline{\mathrm{MF}}^{gr}(A) \\
\cup & & \cup \\
\left.\begin{array}{c}\mathrm{Pic}^0(E) \setminus \{\mathscr{O}_E\} \\ \mathrm{Pic}^{-1}(E) \\ \mathrm{Pic}^{-2}(E)\end{array}\right\} & \longmapsto & \text{rank one MFs}
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3. One can reduce the proof to the calculation of the matrix factorisations associated to closed points of E.

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Theorem (Atiyah, Lenzing-Meltzer, ...)

By a successive application of $T_{\mathscr{O}_E}$, $[1]$ *and* $\mathscr{O}_E(e) \otimes -$, *any indecomposable vector bundle can be transformed into a skyscraper sheaf supported at a closed point.*

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Definition

The twist functor $T_{\mathscr{O}_E}$ is defined to be the Fourier-Mukai functor associated to the kernel $\mathscr{O}_{E\times E}(-\Delta)[1] \in \mathscr{D}^b(\mathrm{Coh}(E \times E))$. It is an exact autoequivalence of $\mathscr{D}^b(\mathrm{Coh}(E))$.

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- $T_{\mathscr{O}_E}(\mathscr{O}_E) = \mathscr{O}_E$
- $T_{\mathscr{O}_E}(\mathscr{O}_E(-p)) = \kappa(p)[-1]$
- $T_{\mathscr{O}_E}(\mathscr{O}_E(p)) = \kappa(p)$

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Theorem (Ballard-Favero-Katzarkov)

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There is an isomorphism of functors $\Phi \circ \mathrm{T}_{\mathcal{O}_E} \circ (\mathcal{O}_E(1) \otimes -) \cong (1) \circ \Phi$.

Corollary

For any closed point $p \in E$*, we have*

$$\Phi(\mathcal{O}_E(-p)) = \Phi\big(\kappa(p)\big)[-1](1)$$

and

$$\Phi(\mathcal{O}_E(-e-p)) = \Phi\big(\kappa(-p)\big)(1).$$

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Idea

1. *Use Auslander-Reiten sequences. In* $\mathrm{MCM}^{gr}(A)$ *they take the form*

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2. *There is an equivalence of categories* $\mathrm{Vect}(E) \xrightarrow{\cong} \mathrm{MCM}^{gr}(A)$ *which maps Auslander-Reiten sequences to Auslander-Reiten sequences. The bottom entries of the tubes in* $\mathrm{Vect}(E)$ *can all be found via iterated applications of* $T_{\mathscr{O}_E}$*,* $[1]$ *and* $\mathscr{O}_E(e) \otimes -$ *as we have seen.*

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3. *Any of these steps can be performed by a computer.*

Thank you for your attention!