Triangulated surfaces in triangulated categories

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Outline

Part I - Analog

State sums for associative algebras

Context

- A associative finite dimensional $\mathbf{k}$-algebra
- $E = \{e_1, e_2, \ldots, e_r\}$ chosen basis

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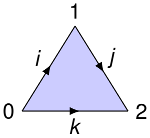
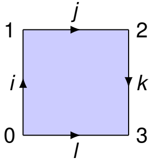
Think of the numbers $\{\lambda_{ij}^k\}$ and $\{\lambda_{ijk}^l\}$ as numerical invariants attached to E-labelled triangles and squares:



$\mapsto \quad \lambda_{ij}^k,$



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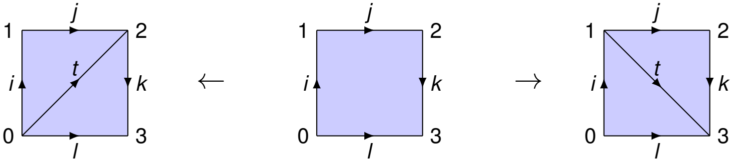
Associativity and triangulations

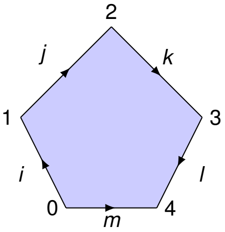
Associativity is then geometrically reflected in terms of the two possible triangulations of the square:



$$\sum_t \lambda_{ij}^{t} \lambda_{tk}^{l} \quad = \quad \lambda_{ijk}^{l} \quad = \quad \sum_t \lambda_{it}^{l} \lambda_{jk}^{t}$$

Polygons

More generally: Numerical invariants for labeled polygons



$\mapsto \quad \lambda^m_{ijkl}$

so that

$$\lambda^m_{ijkl} = \begin{array}{c}\text{explicit formula in terms of } \{\lambda^c_{ab}\} \\ \text{for every triangulation of the pentagon.}\end{array}$$

Frobenius structures

Let $\mathrm{tr} : A \to \mathbf{k}$ be a linear map such that

1. $A \to A^*, a \mapsto \mathrm{tr}(a-)$ is an isomorphism,
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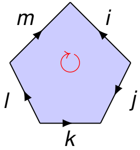
- Dual basis $E^* = \{e_1^*, e_2^*, \ldots, e_r^*\}$ of A with $\operatorname{tr}(e_i e_j^*) = \delta_{ij}$.
- $(E^*)^* = E$ so that $*$ is involutive.
- Our numerical invariants can be computed by the formulas

$$\lambda_{ij}^k = \operatorname{tr}(e_i e_j e_k^*) = \operatorname{tr}(e_j e_k^* e_i) = \operatorname{tr}(e_k^* e_i e_j) \qquad \circlearrowright \mathbb{Z}/(3)$$
$$\lambda_{ijk}^l = \operatorname{tr}(e_i e_j e_k e_l^*) \qquad \circlearrowright \mathbb{Z}/(4)$$
$$\vdots$$

which have a manifest **cyclic symmetry**.

Oriented polygons

Enlarge range of definition of our invariants to **oriented** polygons with **oriented** E-labeled edges:

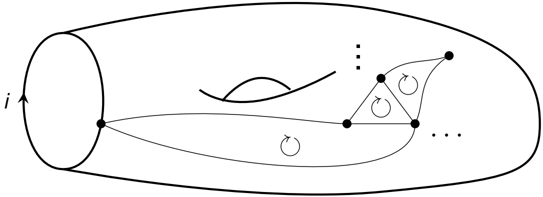


$$\mapsto \quad \mathrm{tr}(e_m e_i^* e_j e_k^* e_l)$$

which can still be computed in terms of a chosen triangulation involving the vertices of the polygon.

Oriented marked surfaces

Natural continuation: Given an compact oriented surface S with a finite nonempty set $M \subset S$ of marked points, we can associate to any Frobenius algebra A, a numerical invariant



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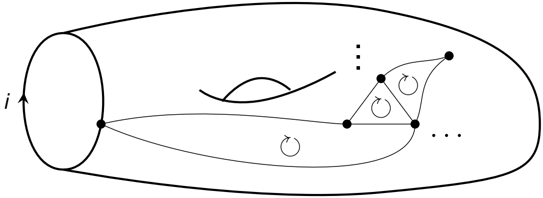

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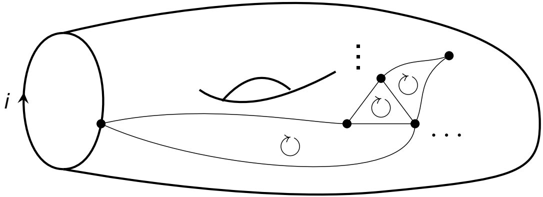

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This really works. . .

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Example

For a finite group G consider the group algebra $\mathbb{C}G$ with its standard Frobenius structure and basis $E = \{g_1, \ldots, g_r\}$. Given a closed oriented marked surface (S, M), we have

$$\lambda_{(S,M)} = \#\,\mathrm{Fun}(\Pi(S, M), BG).$$

Conclusion:

- Associativity
  - $\rightsquigarrow$ the state sum is independent of chosen triangulation
- Frobenius structure
  - $\rightsquigarrow$ cyclic symmetry which nicely interacts with orientation

$\Rightarrow$ numerical invariants of oriented marked surfaces

Part II - Heuristics

State sums for triangulated categories

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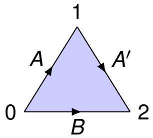

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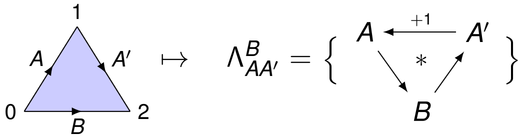

the collection of all distinguished triangles in $\mathcal{T}$ involving the objects determined by the edge labels.

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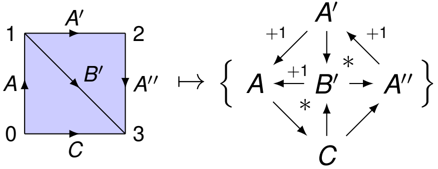

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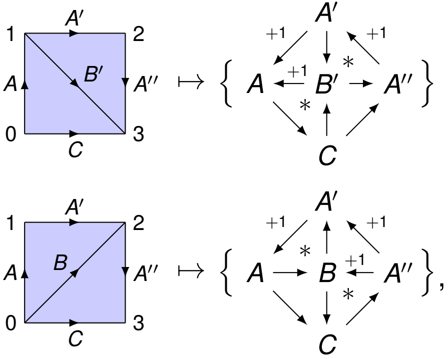

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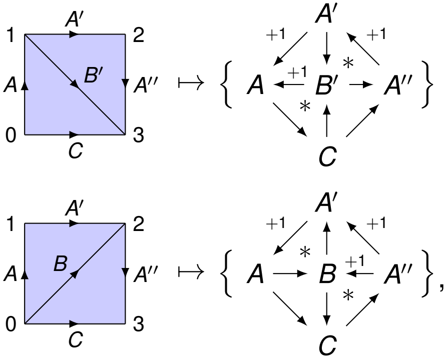

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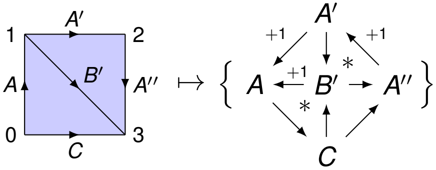

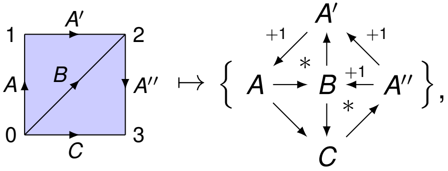

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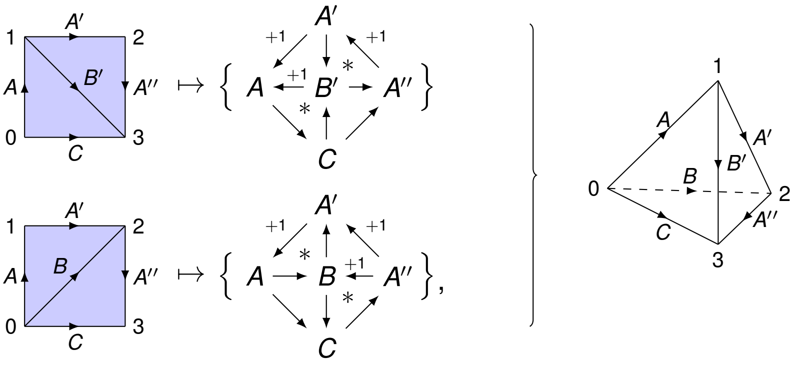

⇝ these diagrams form the upper and lower cap of an octahedron . . . or dually the front and back of a 3-simplex

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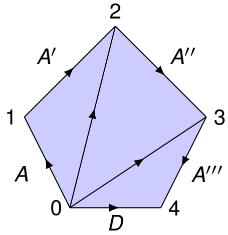
Postnikov systems

More generally, to an E-labeled polygon with a chosen triangulation we associate the collection of certain *Postnikov systems* [Gelfand-Manin] such as



$$\mapsto \left\{ \begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ {\scriptstyle +1}\nwarrow & * & \swarrow \; {\scriptstyle +1}\nwarrow & * & \swarrow \; {\scriptstyle +1}\nwarrow & * & \swarrow \\ & A' & \xleftarrow{+1} & A'' & \xleftarrow{+1} & A''' & \end{array} \right\}.$$

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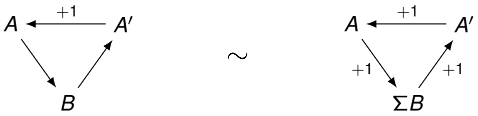
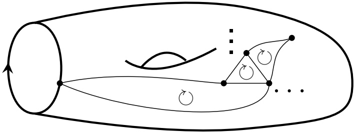


$\rightsquigarrow$ the right-hand form has a manifest **cyclic symmetry** analogous to the symmetry of the trace expression $\mathrm{tr}(e_i e_j e_k^*)$ of a Frobenius algebra

Conclusion

This heuristics suggests:

1. Existence of an invariant $\Lambda_{(S,M)}$ of a marked compact oriented surface (S, M) associated with any 2-periodic triangulated category $\mathcal{T}$:



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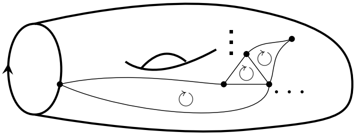

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2. State sum formulas should lead to a description of this invariants in terms of

$$\left\{ \begin{array}{c} \text{collections of distinguished triangles in } \mathcal{T} \text{ parametrized} \\ \text{by a chosen triangulation } \Delta(S, M) \text{ of the surface.} \end{array} \right\}$$

…let's call these collections **surface Postnikov systems.**

Part III - Results

The theory of cyclic 2-Segal spaces

Waldhausen $S_\bullet$-construction

$\mathcal{T}$ triangulated category equipped with differential $\mathbb{Z}/(2)$-graded enhancement.

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$$
\begin{array}{ccccccccc}
0 & & \text{objects} & & \text{triangles} & & \text{octahedral diagrams} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
S_0(\mathcal{T}) & \leftleftarrows & S_1(\mathcal{T}) & \overset{\longleftarrow}{\leftleftarrows} & S_2(\mathcal{T}) & \overset{\leftleftarrows}{\leftleftarrows} & S_3(\mathcal{T}) & \overset{\leftleftarrows}{\overset{\longleftarrow}{\leftleftarrows}} & \cdots
\end{array}
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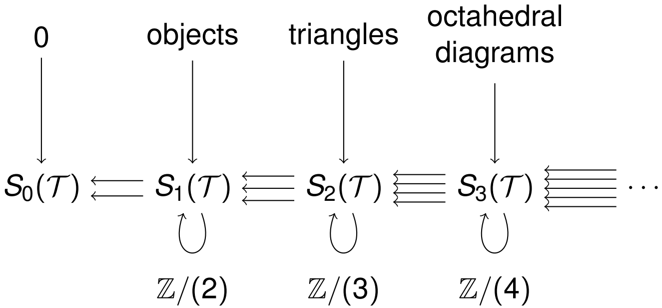


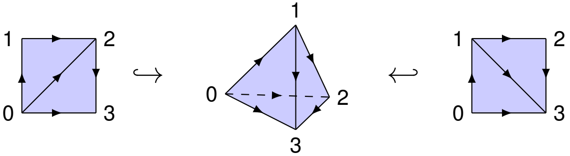
Main result I

Theorem

Let $\mathcal{T}$ be a triangulated category equipped with a differential $\mathbb{Z}/2\mathbb{Z}$-graded enhancement. Denote by $S_\bullet(\mathcal{T})$ the simplicial space given by Waldhausen's $S_\bullet$-construction. Then

1. *$S_\bullet(\mathcal{T})$ is a 2-Segal space ("associativity"),*
2. *$S_\bullet(\mathcal{T})$ is canonically a cyclic space in the sense of Connes ("Frobenius structure").*

Lowest 2-Segal conditions



$$S_2(\mathcal{T}) \times_{S_1(\mathcal{T})} S_2(\mathcal{T}) \xleftarrow{\simeq} S_3(\mathcal{T}) \xrightarrow{\simeq} S_2(\mathcal{T}) \times_{S_1(\mathcal{T})} S_2(\mathcal{T})$$

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such that, for every $\mathcal{T} \in \mathrm{dgcat}^{(2)}$, we have

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Solution

$$\mathcal{F}^n := \mathrm{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$$

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- On the level of homotopy categories [A. Takahashi]:

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- ...this is Happel's root category
- $\mathbb{Z}/(n+1)$-action is given by the Coxeter functor [Bernstein-Gelfand-Ponomarev]

Main Theorem II

Theorem

Let $\mathbf{C}$ be a combinatorial model category and let X be a cyclic 2-Segal object in $\mathbf{C}$. Let (S, M) be a stable compact oriented marked surface. Then there exists an object $X_{(S,M)}$ in $\mathrm{Ho}(\mathbf{C})$ which, for every triangulation $\Delta(S, M)$ of (S, M), comes equipped with canonical isomorphism

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Remark

The isomorphism should be regarded as a **categorified state sum** *which computes* $X_{(S,M)}$ *in terms of a chosen triangulation.*

Are these categorified state sums computable?

Apply the theorem to the universal $S_\bullet$-construction $\mathcal{F}^\bullet$ to form the **universal Postnikov category** of (S, M):

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so that, for any $\mathcal{T} \in \text{dgcat}^{(2)}$, we have

$$\begin{aligned} \text{Map}(\mathcal{F}^{(S,M)}, \mathcal{T}) &\simeq S_\bullet(\mathcal{T})_{(S,M)} \\ &\simeq \{(S, M)\text{-surface Postnikov systems in } \mathcal{T}\}. \end{aligned}$$

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$$\mathcal{F}^{(S,M)} \cong \underset{\Lambda^n \to \Delta(S,M)}{\text{hocolim}} \mathcal{F}^n \quad \in \text{dgcat}^{(2)}$$

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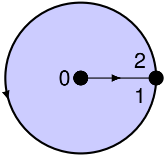
$$\begin{aligned} \text{Map}(\mathcal{F}^{(S,M)}, \mathcal{T}) &\simeq S_\bullet(\mathcal{T})_{(S,M)} \\ &\simeq \{(S, M)\text{-surface Postnikov systems in } \mathcal{T}\}. \end{aligned}$$

The dg category $\mathcal{F}^{(S,M)}$ can be explicitly computed in examples.

Part IV - Examples

Example 1: Disk with two marked points

$(S, M) =$



$\rightsquigarrow \quad \mathcal{F}^{(S,M)} = \mathsf{D}^b(\mathrm{coh}\,\mathbb{A}^1)^{(2)}$

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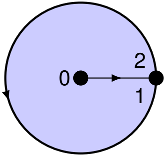
$(S, M) =$ 

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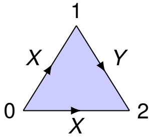
The universal Postnikov system in $\mathcal{F}^{(S,M)}$ is given by



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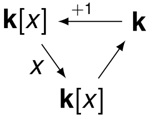

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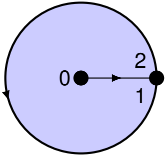
$(S, M) =$ 

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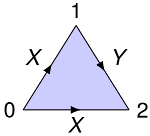

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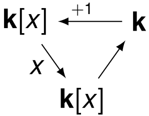
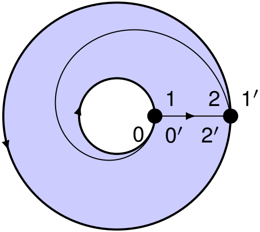


The mapping class group $\mathrm{Mod}(S, M)$ is trivial.

Example 2: Annulus with two marked points

$(S, M) =$ 

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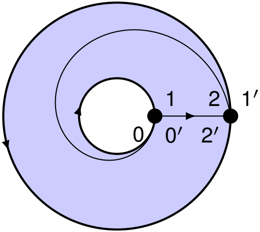

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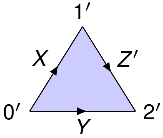

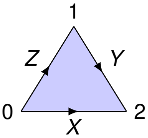

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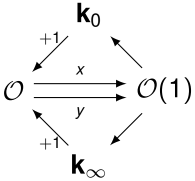

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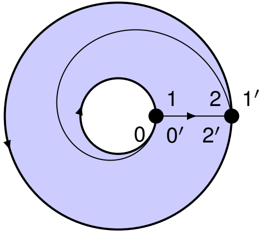

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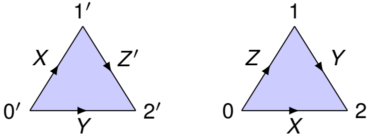

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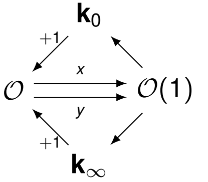
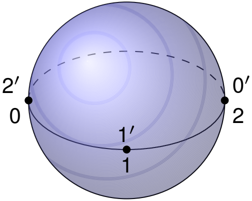


The generator of $\mathrm{Mod}(S, M) \cong \mathbb{Z}$ acts via $- \otimes \mathcal{O}(1)$.

Example 3: Sphere with 3 marked points

$(S, M) =$ 

$$\rightsquigarrow \mathcal{F}^{(S,M)} = \mathrm{D}^b(\mathrm{coh}\, \underbrace{\mathbf{k}[x,y]/(xy)}_{S})^{(2)}$$

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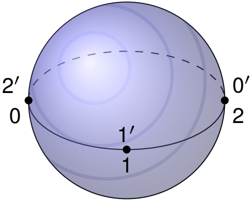
$(S, M) =$  $\rightsquigarrow \mathcal{F}^{(S,M)} = \mathrm{D}^b(\mathrm{coh}\, \underbrace{\mathbf{k}[x,y]/(xy)}_{S})^{(2)}$

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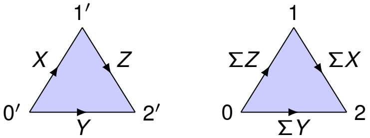

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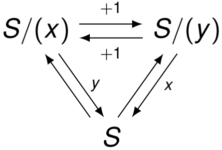

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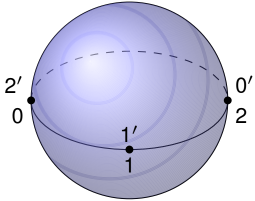

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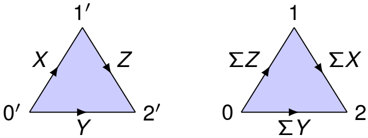

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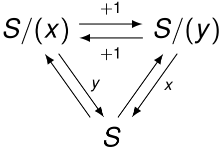

There is an action of $\mathrm{Mod}(S, M) \cong S_3$ on $\mathcal{F}^{(S,M)}$.

Part V - Surprise

Symplectic geometry of homological algebra

This list of examples also appears in the context of a proposal of Maxim Kontsevich:

Symplectic geometry of homological algebra

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Our theory implements a 2-dimensional instance of his general program to localize Fukaya categories of Stein manifolds along singular Lagrangian spines.

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Definition

$\mathcal{F}^{(S,M)}$	topological coFukaya category of (S, M)
$(\mathcal{F}^{(S,M)})^\vee$	topological Fukaya category of (S, M)

Two classes of objects in the topological Fukaya category $(\mathcal{F}^{(S,M)})^\vee$ correspond to:

1. oriented immersed arcs in $S \setminus M$ starting and ending in $\partial S \setminus M$.
2. oriented immersed closed curves in $S \setminus M$ equipped with a flat $\mathbf{k}^*$-principal bundle.

Example 2 - revisited

Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\mathrm{coh}\,\mathbb{P}^1)^{(2)}$.

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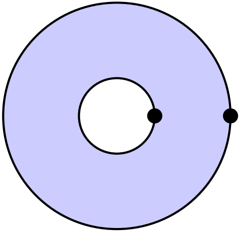

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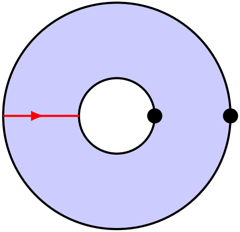
$(S, M) =$ 

$\mathcal{O}$

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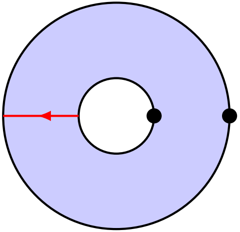
$(S, M) =$ 

$\Sigma\mathcal{O}$

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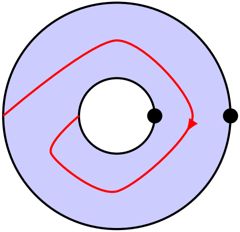
$(S, M) =$ 

$\mathcal{O}(1)$

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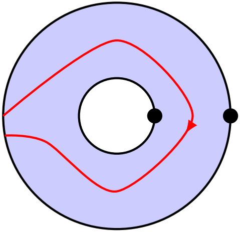
$(S, M) =$ 

$\mathbb{C}_0$

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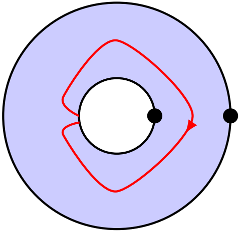
$(S, M) =$ 

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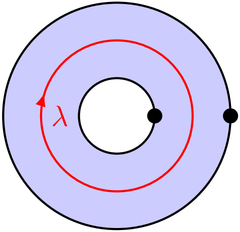
$(S, M) =$ 

$\mathbb{C}_\lambda,\ \lambda \in \mathbb{C}^*$