

Triangulated surfaces in triangulated categories

T. Dyckerhoff

joint work with M. Kapranov

Mathematical Institute
University of Oxford

03/27/14 – Soltau

Outline

- Part I** Analog
- Part II** Heuristics
- Part III** Results
- Part IV** Examples
- Part V** Surprise

Part I - Analog

State sums for associative algebras

Context

- ▶ A associative finite dimensional \mathbf{k} -algebra
- ▶ $E = \{e_1, e_2, \dots, e_r\}$ chosen basis

Context

- ▶ A associative finite dimensional \mathbf{k} -algebra
- ▶ $E = \{e_1, e_2, \dots, e_r\}$ chosen basis
- ▶ Multiplication law is encoded in structure constants: $\lambda_{ij}^k \in \mathbf{k}$
with $e_i e_j = \sum_k \lambda_{ij}^k e_k$

Context

- ▶ A associative finite dimensional \mathbf{k} -algebra
- ▶ $E = \{e_1, e_2, \dots, e_r\}$ chosen basis
- ▶ Multiplication law is encoded in structure constants: $\lambda_{ij}^k \in \mathbf{k}$
with $e_i e_j = \sum_k \lambda_{ij}^k e_k$
- ▶ Associativity is expressed by the equations

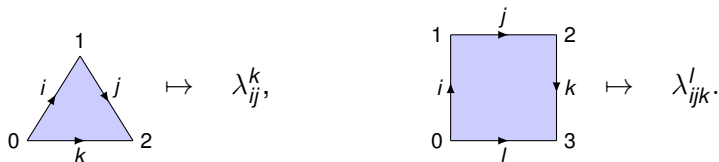
$$\sum_t \lambda_{ij}^t \lambda_{tk}^l = \lambda_{ijk}^l = \sum_t \lambda_{it}^l \lambda_{jk}^t$$

Context

- ▶ A associative finite dimensional \mathbf{k} -algebra
- ▶ $E = \{e_1, e_2, \dots, e_r\}$ chosen basis
- ▶ Multiplication law is encoded in structure constants: $\lambda_{ij}^k \in \mathbf{k}$ with $e_i e_j = \sum_k \lambda_{ij}^k e_k$
- ▶ Associativity is expressed by the equations

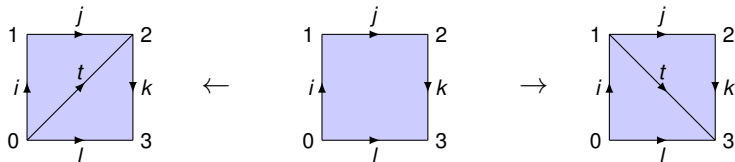
$$\sum_t \lambda_{ij}^t \lambda_{tk}^l = \lambda_{ijk}^l = \sum_t \lambda_{it}^l \lambda_{jk}^t$$

Think of the numbers $\{\lambda_{ij}^k\}$ and $\{\lambda_{ijk}^l\}$ as numerical invariants attached to E -labelled triangles and squares:



Associativity and triangulations

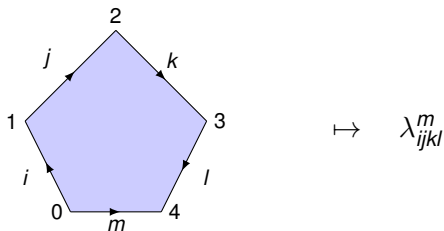
Associativity is then geometrically reflected in terms of the two possible triangulations of the square:



$$\sum_t \lambda_{ij}^t \lambda_{tk}^l = \lambda_{ijk}^l = \sum_t \lambda_{it}^l \lambda_{jk}^t$$

Polygons

More generally: Numerical invariants for labeled polygons



so that

$\lambda_{ijkl}^m =$ explicit formula in terms of $\{\lambda_{ab}^c\}$
for every triangulation of the pentagon.

Frobenius structures

Let $\text{tr} : A \rightarrow \mathbf{k}$ be a linear map such that

1. $A \rightarrow A^*$, $a \mapsto \text{tr}(a-)$ is an isomorphism,
2. $\text{tr}(ab) = \text{tr}(ba)$.

Frobenius structures

Let $\text{tr} : A \rightarrow \mathbf{k}$ be a linear map such that

1. $A \rightarrow A^*$, $a \mapsto \text{tr}(a-)$ is an isomorphism,
2. $\text{tr}(ab) = \text{tr}(ba)$.

Then

- ▶ Dual basis $E^* = \{e_1^*, e_2^*, \dots, e_r^*\}$ of A with $\text{tr}(e_i e_j^*) = \delta_{ij}$.

Frobenius structures

Let $\text{tr} : A \rightarrow \mathbf{k}$ be a linear map such that

1. $A \rightarrow A^*$, $a \mapsto \text{tr}(a-)$ is an isomorphism,
2. $\text{tr}(ab) = \text{tr}(ba)$.

Then

- ▶ Dual basis $E^* = \{e_1^*, e_2^*, \dots, e_r^*\}$ of A with $\text{tr}(e_i e_j^*) = \delta_{ij}$.
- ▶ $(E^*)^* = E$ so that $*$ is involutive.

Frobenius structures

Let $\text{tr} : A \rightarrow \mathbf{k}$ be a linear map such that

1. $A \rightarrow A^*$, $a \mapsto \text{tr}(a-)$ is an isomorphism,
2. $\text{tr}(ab) = \text{tr}(ba)$.

Then

- ▶ Dual basis $E^* = \{e_1^*, e_2^*, \dots, e_r^*\}$ of A with $\text{tr}(e_i e_j^*) = \delta_{ij}$.
- ▶ $(E^*)^* = E$ so that $*$ is involutive.
- ▶ Our numerical invariants can be computed by the formulas

$$\lambda_{ij}^k = \text{tr}(e_i e_j e_k^*) = \text{tr}(e_j e_k^* e_i) = \text{tr}(e_k^* e_i e_j) \quad \circlearrowleft \mathbb{Z}/(3)$$

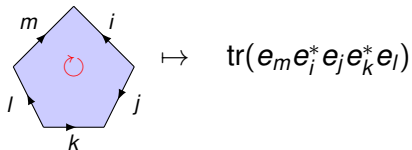
$$\lambda_{ijk}^l = \text{tr}(e_i e_j e_k e_l^*) \quad \circlearrowleft \mathbb{Z}/(4)$$

\vdots

which have a manifest **cyclic symmetry**.

Oriented polygons

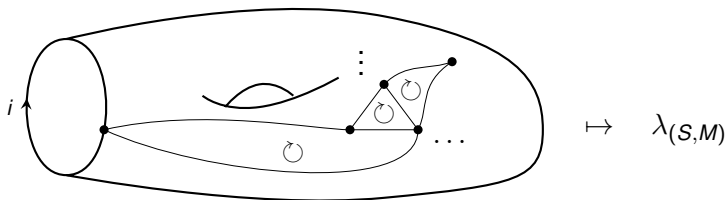
Enlarge range of definition of our invariants to **oriented** polygons with **oriented** E -labeled edges:



which can still be computed in terms of a chosen triangulation involving the vertices of the polygon.

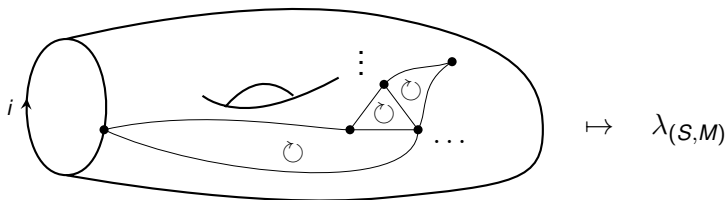
Oriented marked surfaces

Natural continuation: Given an compact oriented surface S with a finite nonempty set $M \subset S$ of marked points, we can associate to any Frobenius algebra A , a numerical invariant



Oriented marked surfaces

Natural continuation: Given an compact oriented surface S with a finite nonempty set $M \subset S$ of marked points, we can associate to any Frobenius algebra A , a numerical invariant

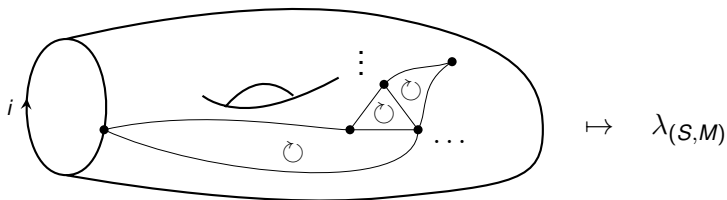


with

$\lambda_{(S,M)}$ = explicit formula for every triangulation of S involving the vertices M .

Oriented marked surfaces

Natural continuation: Given an compact oriented surface S with a finite nonempty set $M \subset S$ of marked points, we can associate to any Frobenius algebra A , a numerical invariant



with

$\lambda_{(S,M)}$ = explicit formula for every triangulation
of S involving the vertices M .

This really works...

Topological field theories

These invariants exist and originate in physics where they are called **partition functions** which we have computed via **state sums**.

Topological field theories

These invariants exist and originate in physics where they are called **partition functions** which we have computed via **state sums**.

The collection of all partition functions associated to a given Frobenius algebra is called a **topological field theory** [Witten, Atiyah, Moore-Segal, ...].

Topological field theories

These invariants exist and originate in physics where they are called **partition functions** which we have computed via **state sums**.

The collection of all partition functions associated to a given Frobenius algebra is called a **topological field theory** [Witten, Atiyah, Moore-Segal, ...].

Example

For a finite group G consider the group algebra $\mathbb{C}G$ with its standard Frobenius structure and basis $E = \{g_1, \dots, g_r\}$. Given a closed oriented marked surface (S, M) , we have

$$\lambda_{(S,M)} = \# \text{Fun}(\Pi(S, M), BG).$$

Conclusion:

- ▶ **Associativity**
 - ↪ the state sum is independent of chosen triangulation
 - ▶ **Frobenius structure**
 - ↪ cyclic symmetry which nicely interacts with orientation
- ⇒ numerical invariants of oriented marked surfaces

Part II - Heuristics

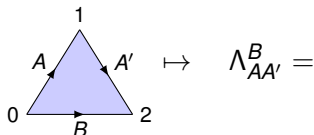
State sums for triangulated categories

Context

- ▶ \mathcal{T} triangulated category
- ▶ $E = \{A, B, \dots, A', B', \dots\}$ set of all objects

Context

- ▶ \mathcal{T} triangulated category
- ▶ $E = \{A, B, \dots, A', B', \dots\}$ set of all objects
- ▶ Associate to an E -labeled triangle



Context

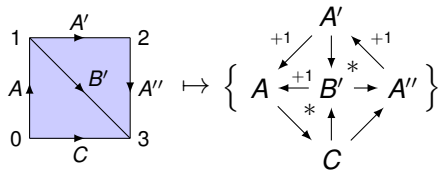
- ▶ \mathcal{T} triangulated category
- ▶ $E = \{A, B, \dots, A', B', \dots\}$ set of all objects
- ▶ Associate to an E -labeled triangle

The diagram illustrates the mapping from an E -labeled triangle to a collection of distinguished triangles. On the left, a blue triangle has vertices labeled 0, 1, and 2. The edges are labeled with objects: the left edge from 0 to 1 is labeled A , the right edge from 1 to 2 is labeled A' , and the bottom edge from 0 to 2 is labeled B . An arrow points to the right, where the expression $\Lambda_{AA'}^B = \left\{ \begin{array}{ccc} A & \xleftarrow{+1} & A' \\ & \searrow * & \nearrow \\ & B & \end{array} \right\}$ is shown. This represents the collection of all distinguished triangles in \mathcal{T} involving the objects A , A' , and B .

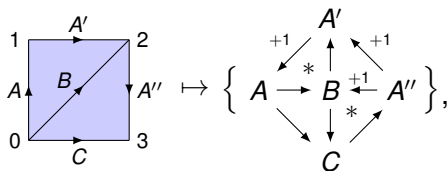
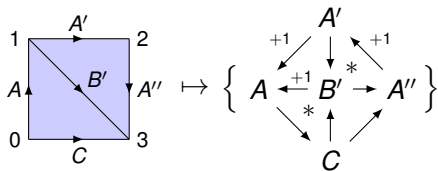
the collection of all distinguished triangles in \mathcal{T} involving the objects determined by the edge labels.

To a triangulated square with E -labeled edges, we attach the following collections of diagrams

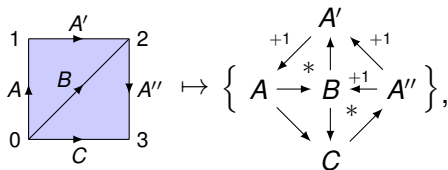
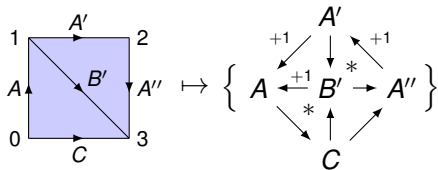
To a triangulated square with E -labeled edges, we attach the following collections of diagrams



To a triangulated square with E -labeled edges, we attach the following collections of diagrams

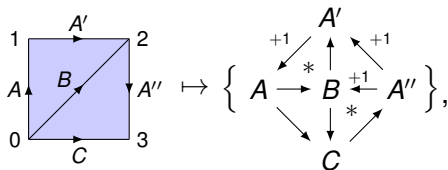
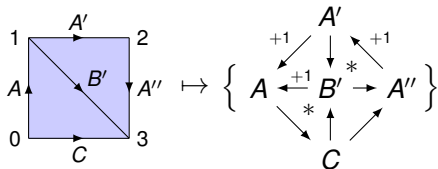


To a triangulated square with E -labeled edges, we attach the following collections of diagrams



\rightsquigarrow these diagrams form the upper and lower cap of an octahedron

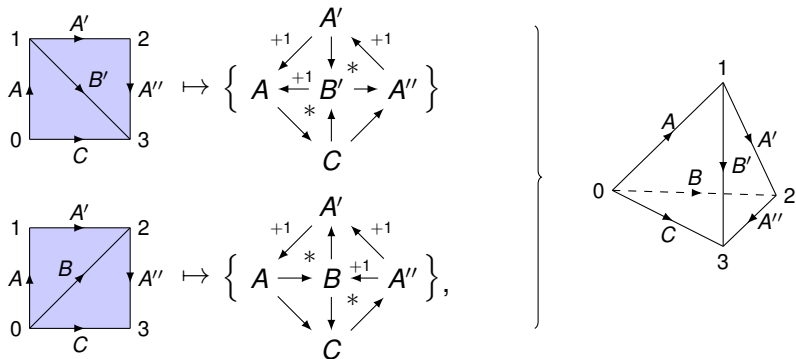
To a triangulated square with E -labeled edges, we attach the following collections of diagrams



↪ these diagrams form the upper and lower cap of an octahedron

Heuristically: The octahedral axiom is the analog of associativity allowing us to pass from one triangulation to another.

To a triangulated square with E -labeled edges, we attach the following collections of diagrams

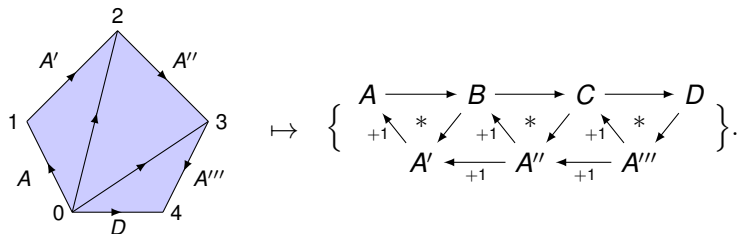


\rightsquigarrow these diagrams form the upper and lower cap of an octahedron ... or dually the front and back of a 3-simplex

Heuristically: The octahedral axiom is the analog of associativity allowing us to pass from one triangulation to another.

Postnikov systems

More generally, to an E -labeled polygon with a chosen triangulation we associate the collection of certain *Postnikov systems* [Gelfand-Manin] such as



What is the analog of a Frobenius structure?

What is the analog of a Frobenius structure?

Answer

Assume that the suspension functor of \mathcal{T} “is” 2-periodic:

$$\Sigma^2 \cong \text{id}.$$

What is the analog of a Frobenius structure?

Answer

Assume that the suspension functor of \mathcal{T} “is” 2-periodic:

$$\Sigma^2 \cong \text{id}.$$

We rewrite a distinguished triangle as


$$\begin{array}{ccc} A & \xleftarrow{+1} & A' \\ & \searrow & \nearrow \\ & B & \end{array} \quad \sim \quad \begin{array}{ccc} A & \xleftarrow{+1} & A' \\ & \xrightarrow{+1} \searrow & \nearrow +1 \\ & \Sigma B & \end{array}$$

What is the analog of a Frobenius structure?

Answer

Assume that the suspension functor of \mathcal{T} “is” 2-periodic:

$$\Sigma^2 \cong \text{id}.$$

We rewrite a distinguished triangle as

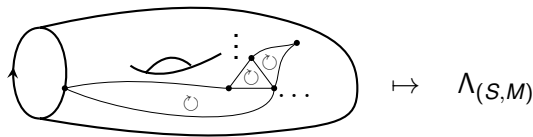
$$\begin{array}{ccc} A & \xleftarrow{+1} & A' \\ & \searrow & \nearrow \\ & B & \end{array} \quad \sim \quad \begin{array}{ccc} A & \xleftarrow{+1} & A' \\ & \xrightarrow{+1} \searrow & \nearrow +1 \\ & \Sigma B & \end{array}$$

\rightsquigarrow the right-hand form has a manifest **cyclic symmetry** analogous to the symmetry of the trace expression $\text{tr}(e_i e_j e_k^*)$ of a Frobenius algebra

Conclusion

This heuristics suggests:

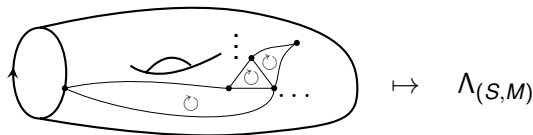
1. Existence of an invariant $\Lambda_{(S,M)}$ of a marked compact oriented surface (S, M) associated with any 2-periodic triangulated category \mathcal{T} :



Conclusion

This heuristics suggests:

1. Existence of an invariant $\Lambda_{(S,M)}$ of a marked compact oriented surface (S, M) associated with any 2-periodic triangulated category \mathcal{T} :



2. State sum formulas should lead to a description of this invariants in terms of

{ collections of distinguished triangles in \mathcal{T} parametrized
by a chosen triangulation $\Delta(S, M)$ of the surface. }

... let's call these collections **surface Postnikov systems**.

Part III - Results

The theory of cyclic 2-Segal spaces

Waldhausen S_\bullet -construction

\mathcal{T} triangulated category equipped with differential $\mathbb{Z}/(2)$ -graded enhancement.

Waldhausen S_\bullet -construction

\mathcal{T} triangulated category equipped with differential $\mathbb{Z}/(2)$ -graded enhancement.

The Waldhausen S_\bullet -construction $S_\bullet(\mathcal{T})$ is the simplicial space:

Waldhausen S_\bullet -construction

\mathcal{T} triangulated category equipped with differential $\mathbb{Z}/(2)$ -graded enhancement.

The Waldhausen S_\bullet -construction $S_\bullet(\mathcal{T})$ is the simplicial space:

$$\begin{array}{ccccccc} & 0 & & \text{objects} & & \text{triangles} & & \text{octahedral} \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S_0(\mathcal{T}) & \longleftarrow & S_1(\mathcal{T}) & \longleftarrow & S_2(\mathcal{T}) & \longleftarrow & S_3(\mathcal{T}) & \longleftarrow \cdots \end{array}$$

Waldhausen S_\bullet -construction

\mathcal{T} triangulated category equipped with differential $\mathbb{Z}/(2)$ -graded enhancement.

The Waldhausen S_\bullet -construction $S_\bullet(\mathcal{T})$ is the simplicial space:

$$\begin{array}{ccccccc} & & 0 & & \text{objects} & & \text{triangles} & & \text{octahedral} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\ & & S_0(\mathcal{T}) & \longleftarrow & S_1(\mathcal{T}) & \longleftarrow & S_2(\mathcal{T}) & \longleftarrow & S_3(\mathcal{T}) & \longleftarrow \cdots \\ & & & & \uparrow & & \uparrow & & \uparrow & \\ & & & & \mathbb{Z}/(2) & & \mathbb{Z}/(3) & & \mathbb{Z}/(4) & \end{array}$$

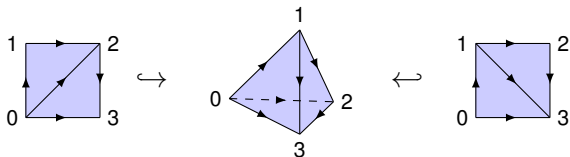
Main result I

Theorem

Let \mathcal{T} be a triangulated category equipped with a differential $\mathbb{Z}/2\mathbb{Z}$ -graded enhancement. Denote by $S_{\bullet}(\mathcal{T})$ the simplicial space given by Waldhausen's S_{\bullet} -construction. Then

- 1. $S_{\bullet}(\mathcal{T})$ is a 2-Segal space (“associativity”),*
- 2. $S_{\bullet}(\mathcal{T})$ is canonically a cyclic space in the sense of Connes (“Frobenius structure”).*

Lowest 2-Segal conditions



$$\mathcal{S}_2(\mathcal{T}) \times_{\mathcal{S}_1(\mathcal{T})} \mathcal{S}_2(\mathcal{T}) \xleftarrow{\cong} \mathcal{S}_3(\mathcal{T}) \xrightarrow{\cong} \mathcal{S}_2(\mathcal{T}) \times_{\mathcal{S}_1(\mathcal{T})} \mathcal{S}_2(\mathcal{T})$$

Proof strategy to exhibit cyclic symmetry

Proof strategy to exhibit cyclic symmetry

Step 1. Find the **universal S_\bullet -construction** in $\text{dgcats}^{(2)}$:

Proof strategy to exhibit cyclic symmetry

Step 1. Find the **universal S_{\bullet} -construction** in $\mathrm{dgc}at^{(2)}$:
cosimplicial differential $\mathbb{Z}/(2)$ -graded category

$$\mathcal{F}^{\bullet} : \Delta \longrightarrow \mathrm{dgc}at^{(2)}$$

such that, for every $\mathcal{T} \in \mathrm{dgc}at^{(2)}$, we have

$$S_{\bullet}(\mathcal{T}) = \mathrm{Map}(\mathcal{F}^{\bullet}, \mathcal{T}).$$

Proof strategy to exhibit cyclic symmetry

Step 1. Find the **universal S_{\bullet} -construction** in $\text{dgcats}^{(2)}$:
cosimplicial differential $\mathbb{Z}/(2)$ -graded category

$$\mathcal{F}^{\bullet} : \Delta \longrightarrow \text{dgcats}^{(2)}$$

such that, for every $\mathcal{T} \in \text{dgcats}^{(2)}$, we have

$$S_{\bullet}(\mathcal{T}) = \text{Map}(\mathcal{F}^{\bullet}, \mathcal{T}).$$

Step 2. Try to find \mathcal{F}^{\bullet} which has manifest cyclic symmetry.

Proof strategy to exhibit cyclic symmetry

Step 1. Find the **universal S_\bullet -construction** in $\text{dgcats}^{(2)}$:
cosimplicial differential $\mathbb{Z}/(2)$ -graded category

$$\mathcal{F}^\bullet : \Delta \longrightarrow \text{dgcats}^{(2)}$$

such that, for every $\mathcal{T} \in \text{dgcats}^{(2)}$, we have

$$S_\bullet(\mathcal{T}) = \text{Map}(\mathcal{F}^\bullet, \mathcal{T}).$$

Step 2. Try to find \mathcal{F}^\bullet which has manifest cyclic symmetry.

Solution

$$\mathcal{F}^n := \text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$$

$\mathbf{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$

- ▶ $\mathbb{Z}/(n+1)$ acts by grading shift (strictly via dg functors)

$\text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$

- ▶ $\mathbb{Z}/(n+1)$ acts by grading shift (strictly via dg functors)
- ▶ Indecomposable objects given by

$$[i, j] = \mathbf{k}[z](i) \begin{array}{c} \xrightarrow{z^{j-i}} \\ \xleftarrow{z^{i-j}} \end{array} \mathbf{k}[z](j)$$

$\text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$

- ▶ $\mathbb{Z}/(n+1)$ acts by grading shift (strictly via dg functors)
- ▶ Indecomposable objects given by

$$[i, j] = \mathbf{k}[z](i) \begin{array}{c} \xrightarrow{z^{j-i}} \\ \xleftarrow{z^{i-j}} \end{array} \mathbf{k}[z](j)$$

- ▶ ... these correspond to the A_n -roots

$\text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$

- ▶ $\mathbb{Z}/(n+1)$ acts by grading shift (strictly via dg functors)
- ▶ Indecomposable objects given by

$$[i, j] = \mathbf{k}[z](i) \begin{array}{c} \xrightarrow{z^{j-i}} \\ \xleftarrow{z^{i-j}} \end{array} \mathbf{k}[z](j)$$

- ▶ ... these correspond to the A_n -roots
- ▶ On the level of homotopy categories [A. Takahashi]:

$$H^0(\text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})) \simeq D^b(A_n - \text{mod})^{(2)}$$

$\text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})$

- ▶ $\mathbb{Z}/(n+1)$ acts by grading shift (strictly via dg functors)
- ▶ Indecomposable objects given by

$$[i, j] = \mathbf{k}[z](i) \begin{array}{c} \xrightarrow{z^{j-i}} \\ \xleftarrow{z^{i-j}} \end{array} \mathbf{k}[z](j)$$

- ▶ ... these correspond to the A_n -roots
- ▶ On the level of homotopy categories [A. Takahashi]:

$$H^0(\text{MF}^{\mathbb{Z}/(n+1)}(\mathbf{k}[z], z^{n+1})) \simeq D^b(A_n - \text{mod})^{(2)}$$

- ▶ ... this is Happel's root category
- ▶ $\mathbb{Z}/(n+1)$ -action is given by the Coxeter functor [Bernstein-Gelfand-Ponomarev]

Main Theorem II

Theorem

Let \mathbf{C} be a combinatorial model category and let X be a cyclic 2-Segal object in \mathbf{C} . Let (S, M) be a stable compact oriented marked surface. Then there exists an object $X_{(S, M)}$ in $\text{Ho}(\mathbf{C})$ which, for every triangulation $\Delta(S, M)$ of (S, M) , comes equipped with canonical isomorphism

$$X_{(S, M)} \xrightarrow{\cong} \text{holim}_{\Lambda^n \rightarrow \Delta(S, M)} X_n.$$

Main Theorem II

Theorem

Let \mathbf{C} be a combinatorial model category and let X be a cyclic 2-Segal object in \mathbf{C} . Let (S, M) be a stable compact oriented marked surface. Then there exists an object $X_{(S, M)}$ in $\text{Ho}(\mathbf{C})$ which, for every triangulation $\Delta(S, M)$ of (S, M) , comes equipped with canonical isomorphism

$$X_{(S, M)} \xrightarrow{\cong} \text{holim}_{\Lambda^n \rightarrow \Delta(S, M)} X_n.$$

Further, the mapping class group of (S, M) acts on $X_{(S, M)}$ via automorphisms in $\text{Ho}(\mathbf{C})$.

Main Theorem II

Theorem

Let \mathbf{C} be a combinatorial model category and let X be a cyclic 2-Segal object in \mathbf{C} . Let (S, M) be a stable compact oriented marked surface. Then there exists an object $X_{(S,M)}$ in $\text{Ho}(\mathbf{C})$ which, for every triangulation $\Delta(S, M)$ of (S, M) , comes equipped with canonical isomorphism

$$X_{(S,M)} \xrightarrow{\cong} \text{holim}_{\Lambda^n \rightarrow \Delta(S,M)} X_n.$$

Further, the mapping class group of (S, M) acts on $X_{(S,M)}$ via automorphisms in $\text{Ho}(\mathbf{C})$.

Remark

The isomorphism should be regarded as a **categorified state sum** which computes $X_{(S,M)}$ in terms of a chosen triangulation.

Are these categorified state sums computable?

Apply the theorem to the universal S_\bullet -construction \mathcal{F}^\bullet to form the **universal Postnikov category** of (S, M) :

$$\mathcal{F}^{(S, M)} \cong \operatorname{hocolim}_{\Lambda^n \rightarrow \Delta(S, M)} \mathcal{F}^n \in \operatorname{dgc}at^{(2)}$$

Are these categorified state sums computable?

Apply the theorem to the universal S_\bullet -construction \mathcal{F}^\bullet to form the **universal Postnikov category** of (S, M) :

$$\mathcal{F}^{(S,M)} \cong \operatorname{hocolim}_{\Lambda^n \rightarrow \Delta(S,M)} \mathcal{F}^n \in \operatorname{dgc}at^{(2)}$$

so that, for any $\mathcal{T} \in \operatorname{dgc}at^{(2)}$, we have

$$\begin{aligned} \operatorname{Map}(\mathcal{F}^{(S,M)}, \mathcal{T}) &\simeq S_\bullet(\mathcal{T})_{(S,M)} \\ &\simeq \{(S, M)\text{-surface Postnikov systems in } \mathcal{T}\}. \end{aligned}$$

Are these categorified state sums computable?

Apply the theorem to the universal S_\bullet -construction \mathcal{F}^\bullet to form the **universal Postnikov category** of (S, M) :

$$\mathcal{F}^{(S,M)} \cong \operatorname{hocolim}_{\Lambda^n \rightarrow \Delta(S,M)} \mathcal{F}^n \in \operatorname{dgc}at^{(2)}$$

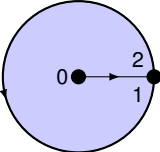
so that, for any $\mathcal{T} \in \operatorname{dgc}at^{(2)}$, we have

$$\begin{aligned} \operatorname{Map}(\mathcal{F}^{(S,M)}, \mathcal{T}) &\simeq S_\bullet(\mathcal{T})_{(S,M)} \\ &\simeq \{(S, M)\text{-surface Postnikov systems in } \mathcal{T}\}. \end{aligned}$$

The dg category $\mathcal{F}^{(S,M)}$ can be explicitly computed in examples.

Part IV - Examples

Example 1: Disk with two marked points

$$(S, M) = \left(\text{Disk with marked points } 0, 1, 2 \right) \rightsquigarrow \mathcal{F}^{(S, M)} = D^b(\text{coh } \mathbb{A}^1)^{(2)}$$
A light blue shaded circle representing a disk. A horizontal line segment is drawn from the center to the right edge. The center is marked with a black dot and labeled '0'. The right edge is marked with a black dot and labeled '1'. A point on the line segment between '0' and '1' is marked with a black dot and labeled '2'. A curved arrow on the left side of the circle indicates a counter-clockwise orientation.

Example 1: Disk with two marked points

$$(S, M) = \text{Disk with two marked points} \rightsquigarrow \mathcal{F}(S, M) = D^b(\text{coh } \mathbb{A}^1)^{(2)}$$

The universal Postnikov system in $\mathcal{F}(S, M)$ is given by

$$\text{Triangle} \rightsquigarrow \text{Commutative Diagram}$$

Example 1: Disk with two marked points

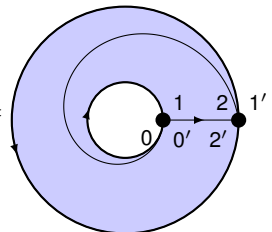
$$(S, M) = \left(\text{Disk with marked points } 0, 1, 2 \right) \rightsquigarrow \mathcal{F}^{(S, M)} = D^b(\text{coh } \mathbb{A}^1)^{(2)}$$

The universal Postnikov system in $\mathcal{F}^{(S, M)}$ is given by

$$\left(\text{Triangle with vertices } 0, 1, 2 \text{ and edges } X, Y \right) \rightsquigarrow \left(\text{Commutative diagram with } \mathbf{k}[x], \mathbf{k}, \mathbf{k}[x] \right)$$

The mapping class group $\text{Mod}(S, M)$ is trivial.

Example 2: Annulus with two marked points

$$(S, M) = \text{Diagram of an annulus with marked points } \rightsquigarrow \mathcal{F}(S, M) = D^b(\text{coh } \mathbb{P}^1)^{(2)}$$
A diagram of an annulus, which is a ring-shaped region between two concentric circles. The region is shaded light blue. On the right side of the annulus, there are two marked points, each represented by a black dot. The inner dot is labeled '1' and the outer dot is labeled '2'. Below these dots, there are labels '0' and '0'' on the left, and '2'' and '1'' on the right. A horizontal line segment with arrows at both ends passes through the dots, with '0' and '0'' on the left side and '2'' and '1'' on the right side. There are also curved arrows on the inner and outer boundaries of the annulus, indicating a direction of flow.

Example 2: Annulus with two marked points

$$(S, M) = \text{Annulus} \rightsquigarrow \mathcal{F}(S, M) = D^b(\text{coh } \mathbb{P}^1)^{(2)}$$

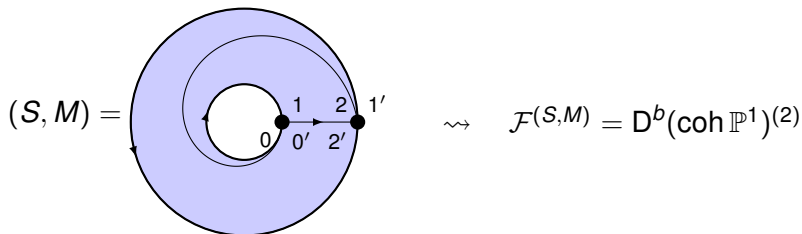
The diagram shows an annulus with two concentric circles. The region between the circles is shaded light blue. Two marked points are shown on the inner boundary: a black dot labeled '0' and another black dot labeled '1'. Corresponding points are shown on the outer boundary: a black dot labeled '0\'' and another black dot labeled '1\''.

The universal Postnikov system in $\mathcal{F}(S, M)$ is given by

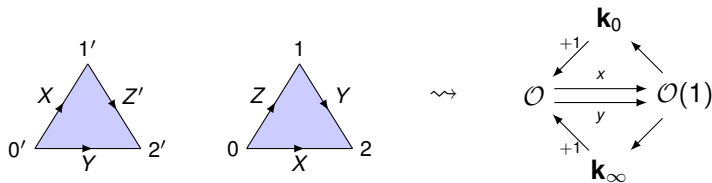
$$\begin{array}{ccc} \begin{array}{c} 1' \\ \swarrow \quad \searrow \\ X \quad \quad Z' \\ \swarrow \quad \searrow \\ 0' \quad \quad 2' \\ \quad \quad \quad \uparrow \\ \quad \quad \quad Y \end{array} & \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ Z \quad \quad Y \\ \swarrow \quad \searrow \\ 0 \quad \quad 2 \\ \quad \quad \quad \uparrow \\ \quad \quad \quad X \end{array} & \rightsquigarrow \begin{array}{ccc} & \mathbf{k}_0 & \\ & \swarrow \quad \searrow & \\ +1 & & \\ \swarrow & & \swarrow \\ \mathcal{O} & \xrightarrow{x} & \mathcal{O}(1) \\ \searrow & & \searrow \\ y & & \\ \swarrow & & \swarrow \\ +1 & & \\ & \mathbf{k}_\infty & \end{array} \end{array}$$

The diagram illustrates the universal Postnikov system. On the left, there are two triangles. The first triangle has vertices labeled 0', 1', and 2'. Arrows point from 0' to 1' (labeled X), from 1' to 2' (labeled Z'), and from 0' to 2' (labeled Y). The second triangle has vertices labeled 0, 1, and 2. Arrows point from 0 to 1 (labeled Z), from 1 to 2 (labeled Y), and from 0 to 2 (labeled X). On the right, a commutative diagram shows the relationship between sheaves. The top node is \mathbf{k}_0 and the bottom node is \mathbf{k}_∞ . The left node is \mathcal{O} and the right node is $\mathcal{O}(1)$. Arrows connect \mathcal{O} to \mathbf{k}_0 (labeled +1), \mathcal{O} to $\mathcal{O}(1)$ (labeled x), $\mathcal{O}(1)$ to \mathbf{k}_∞ (labeled +1), and \mathcal{O} to \mathbf{k}_∞ (labeled y).

Example 2: Annulus with two marked points

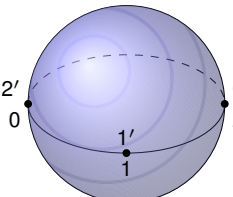


The universal Postnikov system in $\mathcal{F}(S, M)$ is given by



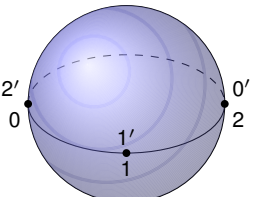
The generator of $\text{Mod}(S, M) \cong \mathbb{Z}$ acts via $- \otimes \mathcal{O}(1)$.

Example 3: Sphere with 3 marked points



$(S, M) = \begin{matrix} 2' \\ 0 \end{matrix} \begin{matrix} 0' \\ 2 \end{matrix} \begin{matrix} 1' \\ 1 \end{matrix} \rightsquigarrow \mathcal{F}(S, M) = D^b(\underbrace{\text{coh } \mathbf{k}[x, y]/(xy)}_S)^{(2)}$

Example 3: Sphere with 3 marked points

$$(S, M) = \begin{array}{c} 2' \\ \bullet \\ 0 \end{array} \begin{array}{c} \bullet \\ 0' \\ 2 \end{array} \begin{array}{c} 1' \\ \bullet \\ 1 \end{array} \rightsquigarrow \mathcal{F}(S, M) = D^b(\text{coh } \underbrace{\mathbf{k}[x, y]/(xy)}_S)^{(2)}$$


The universal Postnikov system is given by

$$\begin{array}{c} 1' \\ \triangle \\ \begin{array}{ccc} X & & Z \\ \swarrow & & \searrow \\ 0' & & 2' \\ \downarrow Y & & \end{array} \end{array} \quad \begin{array}{c} 1 \\ \triangle \\ \begin{array}{ccc} \Sigma Z & & \Sigma X \\ \swarrow & & \searrow \\ 0 & & 2 \\ \downarrow \Sigma Y & & \end{array} \end{array} \rightsquigarrow \begin{array}{ccc} & \xrightarrow{+1} & \\ S/(x) & \xleftrightarrow{+1} & S/(y) \\ & \swarrow y & \searrow x \\ & S & \end{array}$$

Example 3: Sphere with 3 marked points

$$(S, M) = \begin{array}{c} 2' \\ \bullet \\ 0 \end{array} \begin{array}{c} \text{Sphere} \\ \text{with 3 marked points} \\ \begin{array}{c} 0' \\ \bullet \\ 2 \\ \bullet \\ 1' \\ \bullet \\ 1 \end{array} \end{array} \rightsquigarrow \mathcal{F}(S, M) = D^b(\text{coh } \underbrace{\mathbf{k}[x, y]/(xy)}_S)^{(2)}$$

The universal Postnikov system is given by

$$\begin{array}{c} 1' \\ \triangle \\ \begin{array}{ccc} X & & Z \\ \swarrow & & \searrow \\ 0' & & 2' \\ \downarrow Y & & \end{array} \end{array} \quad \begin{array}{c} 1 \\ \triangle \\ \begin{array}{ccc} \Sigma Z & & \Sigma X \\ \swarrow & & \searrow \\ 0 & & 2 \\ \downarrow \Sigma Y & & \end{array} \end{array} \rightsquigarrow \begin{array}{ccc} & \xrightarrow{+1} & \\ S/(x) & \xleftrightarrow{+1} & S/(y) \\ & \swarrow y & \searrow x \\ & S & \end{array}$$

There is an action of $\text{Mod}(S, M) \cong S_3$ on $\mathcal{F}(S, M)$.

Part V - Surprise

Symplectic geometry of homological algebra

This list of examples also appears in the context of a proposal of Maxim Kontsevich:

Symplectic geometry of homological algebra

This list of examples also appears in the context of a proposal of Maxim Kontsevich:

Our theory implements a 2-dimensional instance of his general program to localize Fukaya categories of Stein manifolds along singular Lagrangian spines.

Symplectic geometry of homological algebra

This list of examples also appears in the context of a proposal of Maxim Kontsevich:

Our theory implements a 2-dimensional instance of his general program to localize Fukaya categories of Stein manifolds along singular Lagrangian spines.

Definition

$\mathcal{F}^{(S,M)}$	topological coFukaya category of (S, M)
$(\mathcal{F}^{(S,M)})^\vee$	topological Fukaya category of (S, M)

Symplectic geometry of homological algebra

This list of examples also appears in the context of a proposal of Maxim Kontsevich:

Our theory implements a 2-dimensional instance of his general program to localize Fukaya categories of Stein manifolds along singular Lagrangian spines.

Definition

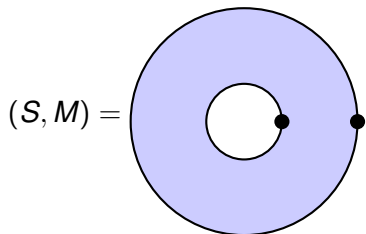
$\mathcal{F}^{(S,M)}$	topological coFukaya category of (S, M)
$(\mathcal{F}^{(S,M)})^\vee$	topological Fukaya category of (S, M)

Two classes of objects in the topological Fukaya category $(\mathcal{F}^{(S,M)})^\vee$ correspond to:

1. oriented immersed arcs in $S \setminus M$ starting and ending in $\partial S \setminus M$.
2. oriented immersed closed curves in $S \setminus M$ equipped with a flat \mathbf{k}^* -principal bundle.

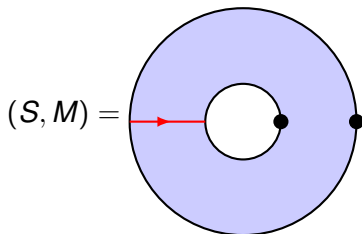
Example 2 - revisited

Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\text{coh } \mathbb{P}^1)^{(2)}$.



Example 2 - revisited

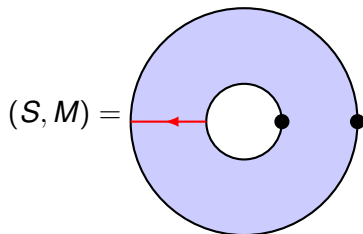
Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\text{coh } \mathbb{P}^1)^{(2)}$.



\mathcal{O}

Example 2 - revisited

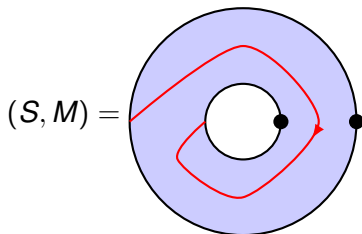
Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\text{coh } \mathbb{P}^1)^{(2)}$.



$\Sigma \mathcal{O}$

Example 2 - revisited

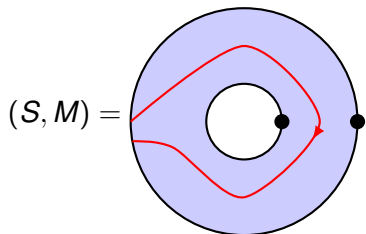
Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\text{coh } \mathbb{P}^1)^{(2)}$.



$\mathcal{O}(1)$

Example 2 - revisited

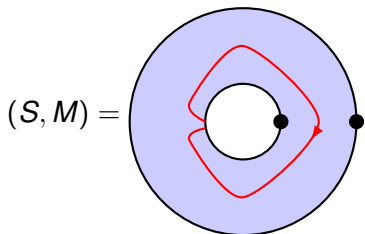
Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\text{coh } \mathbb{P}^1)^{(2)}$.



\mathbb{C}_0

Example 2 - revisited

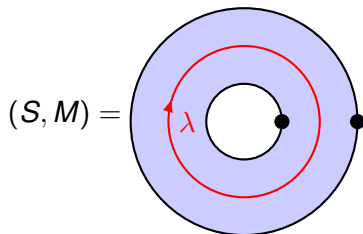
Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\text{coh } \mathbb{P}^1)^{(2)}$.



\mathbb{C}_∞

Example 2 - revisited

Set $\mathbf{k} = \mathbb{C}$. We have $(\mathcal{F}^{(S,M)})^\vee \simeq D^b(\text{coh } \mathbb{P}^1)^{(2)}$.



$$\mathbb{C}_\lambda, \lambda \in \mathbb{C}^*$$