What is a spherical variety?

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Our favorite objects

- $k = \bar{k}$ a field
- G a connected reductive algebraic group
- B a Borel subgroup of G
- X an algebraic variety equipped with an action of G

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A spherical variety is...

... a G-variety X satisfying one of the following equivalent conditions

- Every birational G-equivariant model of X has finitely many G-orbits.
- B has finitely many orbits in X.
- B has an open orbit in X.
- For any G-line bundle $\mathscr{L}$ on X, $H^0(X, \mathscr{L})$ is a multiplicity-free G-module. (if X is quasi-projective)

Obvious example: flag varieties

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Obvious examples of spherical varieties X

- Flag varieties
- Orbit closures in a finite dimensional G-module V of any sum of highest weight vectors of V
- $X = G$ as a $(G \times G)$-variety: $X \simeq G \times G/\mathrm{diag}G$
- $X = \mathbb{P}^1 \times \mathbb{P}^1$ acted on by SL_2 diagonally

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The group G is our model

To any given complex spherical G-variety X, we can attach

- a weight lattice $\Lambda(X)$
- a co-weight lattice $\Lambda(X)^\vee := \mathrm{Hom}(\Lambda, \mathbb{Q})$
- a finitely generated convex cone $\mathcal{V}(X) \subset \Lambda(X)^\vee$
- a finite group W_X, the *Little Weyl group of X*

Theorem (Brion/Knop)

- *The cone $\mathcal{V}(X)$ is a fundamental domain for W_X.*
- *The extremal rays of the dual cone of $\mathcal{V}(X)$ generate a root system: the root system of X.*

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  - positive characteristic
  - real spherical varieties

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