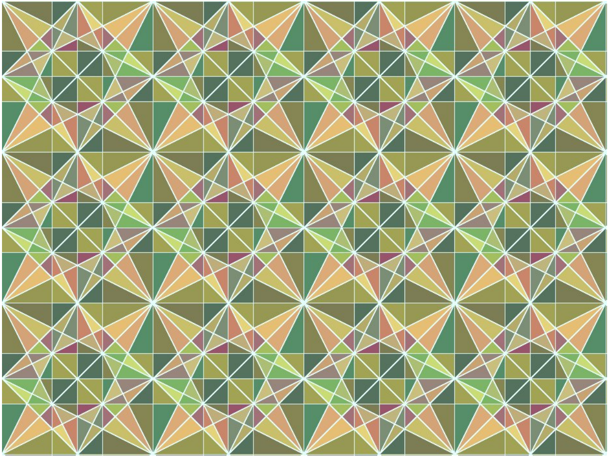

What is a Nichols algebra?

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Braided vector spaces

Definition

Let V be a vector space,

$$c : V \otimes V \to V \otimes V$$

a linear isomorphism with

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

Then c is a *braiding*, and (V, c) is a *braided vector space*.

Nichols algebras

Define a map $\rho : S_n \to \mathrm{End}(V^{\otimes n})$ by:

For a transposition $(i, i+1) \in S_n$ let

$$\rho((i, i+1)) := \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes c \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

where c acts in the copies i and $i+1$ of V.

If $\omega = \tau_1 \dots \tau_\ell$ is a reduced expression of $\omega \in S_n$, then

$$\rho(\omega) := \rho(\tau_1) \dots \rho(\tau_\ell).$$

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Definition

Let $\mathfrak{S}_n := \sum_{\omega \in S_n} \rho(\omega)$.

$$\mathfrak{B}(V) := \bigoplus_{n \geq 0} T^n(V) / \ker(\mathfrak{S}_n)$$

is called the *Nichols algebra* of (V, c).

Examples

- $c(x \otimes y) = y \otimes x \quad$ for all $x, y \in V$:
  $\mathfrak{B}(V) = S(V)$ symmetric algebra

- $c(x \otimes y) = -y \otimes x \quad$ for all $x, y \in V$:
  $\mathfrak{B}(V) = \Lambda(V)$ exterior algebra

Nichols algebras – Motivation

- Nichols (1978): construction of examples of Hopf algebras
- Woronowicz (1988): build a "quantum differential calculus"
- Lusztig (1993), Rosso (1994), Schauenburg (1996): abstract definition of quantized universal enveloping algebras
- Andruskiewitsch-Schneider (1998): essential tool in the classification of pointed Hopf algebras

Nichols algebras – Main problems

Let (V, c) be a braided vector space.

- Is $\mathfrak{B}(V)$ finite dimensional?
- Compute the defining relations of $\mathfrak{B}(V)$.

Examples

Let $A = (a_{ij})_{1\leq i,j\leq r}$ be a Cartan matrix of finite type and $d_1, \ldots, d_r \in \mathbb{N}_{>0}$ be such that $d_i a_{ij} = d_j a_{ji}$.

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Let V be a vector space over $\mathbf{k}$ with basis $x_1, \dots, x_r$, and $q \in \mathbf{k}$, $c : V \otimes V \to V \otimes V$ given by $c(x_i \otimes x_j) = q^{d_i a_{ij}} x_j \otimes x_i$.

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Theorem (Lusztig)

If q is a root of unity of odd order N with $3 \nmid N$, then $\mathfrak{B}(V)$ is finite dimensional with basis [...].

$\mathfrak{B}(V)$ is the "positive part" of the *Frobenius-Lusztig* kernel of the Lie algebra associated to A.

Diagonal type

Definition

$\{x_1, \ldots, x_r\}$ Basis of V,

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad q_{ij} \in \mathbb{C}.$$

Then c and $\mathfrak{B}(V)$ are called of *diagonal type*.

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The numbers q_{ij}, $i, j = 1, \ldots, r$ define a *bicharacter*

$$\chi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{C}, \quad ((a_1, \ldots, a_r), (b_1, \ldots, b_r)) \mapsto \prod_{i,j=1}^{r} q_{ij}^{a_i b_j}.$$

PBW basis for diagonal type

Let (V, c) be of diagonal type.

Theorem (Kharchenko, 1999)

There exists a totally ordered index set $(L, \leq)$ and $\mathbb{Z}^r$-homogeneous elements $X_\ell \in \mathfrak{B}(V)$, $\ell \in L$ such that

$$\{X_{\ell_1}^{m_1} \cdots X_{\ell_\nu}^{m_\nu} \mid \nu \geq 0, \quad \ell_1, \ldots, \ell_\nu \in L, \quad \ell_1 > \ldots > \ell_\nu, \\ 0 \leq m_i < h_{\ell_\nu} \ \forall i = 1, \ldots, \nu\}$$

is a vector space basis of $\mathfrak{B}(V)$, where

$$h_\ell = \min\{m \in \mathbb{N} \mid 1 + q_\ell + \ldots + q_\ell^{m-1} = 0\} \cup \{\infty\}$$

and $q_\ell = \chi(\deg X_\ell, \deg X_\ell)$, $\ell \in L$.

Finite dimensional Nichols algebras of diagonal type

Theorem (Heckenberger, 2006)

Let $\mathfrak{B}$ be a finite dimensional Nichols algebra of diagonal type.

Let R_+ be the set of degrees of the PBW generators of $\mathfrak{B}$.
Then $R_+ \cup -R_+$ is a root system of a finite Weyl groupoid.

Result (Angiono, 2013)

Explicit list of defining relations of a Nichols algebra of diagonal type with finite root system.

Yetter-Drinfeld modules

Definition

Let H be a Hopf algebra and V a module and a comodule over H. Then V is called a *Yetter-Drinfeld module* if

$$\delta_V(hv) = h_1 v_{-1} S(h_3) \otimes h_2 v_0 \quad \forall h \in H, v \in V.$$

A Yetter-Drinfeld module V is a braided vector space via

$$c : V \otimes V \to V \otimes V, \quad v \otimes w \mapsto v_{-1} w \otimes v_0.$$

Example

G a finite group, $H = \mathbb{C}G \Rightarrow$
Yetter-Drinfeld modules are representations of the quantum double $D(G)$.

Yetter-Drinfeld modules

Let V be a Yetter-Drinfeld module over $\mathbb{C}G$ where G is a finite group.

- G abelian $\Rightarrow \mathfrak{B}(V)$ of diagonal type.
- G non-abelian, V irreducible $\Rightarrow \mathfrak{B}(V)$ Nichols algebra of a rack.

A Cartan scheme