Permutation modules for Brauer algebras

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March 26, 2014

Permutation modules for $k\Sigma_r$

Let k be an algebraically closed field and Σ_r the symmetric group on r letters.

Definition

A $k\Sigma_r$-module M is called **permutation module** *if*

- *M is k-free*
- *Σ_r acts as a permutation group on a k-basis of M*

We are interested in the permutation modules of the form

$$M^\lambda := k \bigotimes_{k\Sigma_\lambda} k\Sigma_r$$

where $\Sigma_\lambda = \Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \ldots \times \Sigma_{\lambda_n} \subseteq \Sigma_r$ for a partition λ of r (short: $\lambda \vdash r$).

Young modules

Definition

Let $\lambda \vdash r$ and S_λ the (dual) Specht module with respect to λ. The **Young module** *Y^λ is the unique indecomposable summand of M^λ with a quotient isomorphic to S_λ.*

Theorem (James, 1983)

$$M^\lambda = \bigoplus_{\mu \vdash r} a_\mu Y^\mu, \text{ with } a_\mu = 0 \text{ if } \mu \not\trianglerighteq \lambda.$$

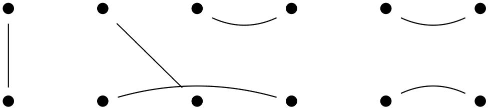
Theorem (Hemmer-Nakano,2004)

$Ext^1_{k\Sigma_r}(Y^\lambda, S_\mu) = 0$ for all $\lambda, \mu \vdash r$, i.e. Y^λ is **relative projective** *with respect to the category of Specht filtered $k\Sigma_r$-modules.*

Brauer algebra

The Brauer algebra $B_k(r, \delta)$ is the associative k-algebra with k-basis consisting of diagrams

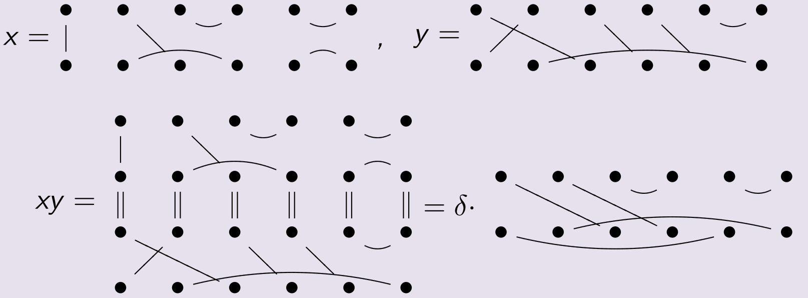
- 2 rows of r dots each
- each dot is connected to exactly one other dot



and multiplication $x \cdot y$:

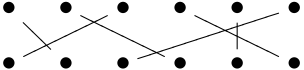
- write x on top of y
- identify bottom row dots of x and top row dots of y
- follow lines from top row dots of x and bottom row dots of y
- replace circles by a factor $\delta \in k$
- remove all inner dots

Example



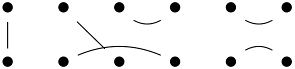
Properties of $B_k(r,\delta)$

- $k\Sigma_r \subset B_k(r,\delta)$ for any parameter $\delta \in k$ e.g.

  $\Sigma_6 \ni (12463) =$ 

- cellularly stratified [Hartmann-Henke-Koenig-Paget, 2010]:
  - iterated inflation of $k\Sigma_l$ along vector spaces V_l

    $A = \bigoplus_{l=1}^{r} k\Sigma_l \otimes_k V_l \otimes_k V_l$ as vector space

     $\leftrightarrow 1 \otimes v \otimes w$ with

    $v =$ 

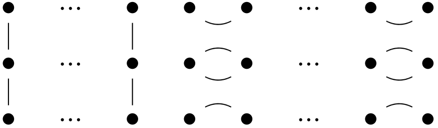
    and $w =$ 
  - $\exists u_l, v_l \in V_l : e_l = 1_{k\Sigma_l} \otimes u_l \otimes v_l$ idempotent
  - $e_l e_m = e_m = e_m e_l$ for $l > m$
- cell modules are of the form $S_\lambda \otimes_k V_l$

Permutation modules for Brauer algebras

Definition (Hartmann-Paget, 2005)

Let $A = B_k(r, \delta)$, λ *a partition of* $l \leq r$, $\frac{r-l}{2} \in \mathbb{N}$ *and*

$e_l := \frac{1}{\delta^{\frac{r-l}{2}}} \cdot$  *if* $\delta \neq 0$

$e_l^2 = e_l$: $\left(\frac{1}{\delta^{\frac{r-l}{2}}}\right)^2 \cdot \delta^{\frac{r-l}{2}} \cdot$ 

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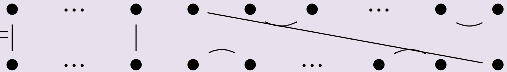
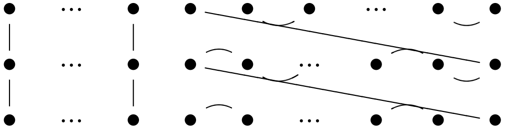
$e_l :=$  *if $\delta = 0$.*

$e_l^2 = e_l$: 

Permutation modules for Brauer algebras

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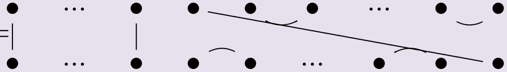
$e_l :=$  *if* $\delta = 0$.

Then the right A-module $M^\lambda \bigotimes_{k\Sigma_l} e_l A$ *is called* **permutation module**.

Proposition (Hartmann-Paget, 2005)

$M^\lambda \underset{k\Sigma_l}{\otimes} e_l A$ has a unique indecomposable direct summand $Y(l, \lambda)$ with quotient isomorphic to $Y^\lambda \otimes_k V_l$.

Theorem (Hartmann-Paget, 2005)

If char$k \neq 2, 3$ then $M^\lambda \underset{k\Sigma_l}{\otimes} e_l A$ is a direct sum of modules $Y(m, \mu)$, where $Y(l, \lambda)$ occurs exactly once and all other summands satisfy $(m, \mu) <' (l, \lambda)$.

Theorem (Hartmann-Paget, 2005)

Let char$k \neq 2, 3$ and $\delta \neq 0$ or $\delta = 0$ and $\lambda \neq \emptyset$. Then $M^\lambda \underset{k\Sigma_l}{\otimes} e_l A$ is relative projective with respect to the category of cell filtered modules.

Generalisation

Let $A = \bigoplus_{l=1}^{r} k\Sigma_l \otimes_k V_l \otimes_k V_l$ be a cellularly stratified algebra such that $k\Sigma_l$ is a subalgebra of $e_l A e_l$ for every l. Set $J_{l-1} := Ae_{l-1}A$.

Definition

$$Ind_l : \text{mod } k\Sigma_l \to \text{mod } A \qquad M \mapsto M \underset{k\Sigma_l}{\otimes} e_l A$$

$$ind_l : \text{mod } k\Sigma_l \to \text{mod } A \qquad M \mapsto M \underset{k\Sigma_l}{\otimes} e_l(A/J_{l-1})$$

$$Res_l : \text{mod } A \to \text{mod } k\Sigma_l \qquad N \mapsto N \underset{A}{\otimes} Ae_l$$

$$res_l : \text{mod } A \to \text{mod } k\Sigma_l \qquad N \mapsto N \underset{A}{\otimes} (A/J_{l-1})e_l$$

Lemma (HHKP,2010)

The cell modules are of the form $ind_l S_\lambda$, where $\lambda \vdash l$.

Definition

The module $Ind_l M^\lambda = M^\lambda \bigotimes_{k\Sigma_l} e_l A$ *is called* **permutation module** *for A.*

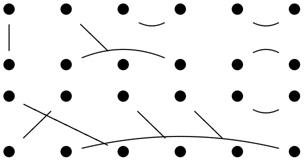
Proposition

If $\operatorname{char} k \neq 2, 3$ *then* $Ind_l M^\lambda$ *has a unique indecomposable direct summand* $Y(l, \lambda)$ *with quotient* $ind_l Y^\lambda$*.*

Assumptions

Let $char k \neq 2, 3$. Assume that A satisfies for $n < l$:

1. $e_l J_n \simeq e_l(J_n/J_{n-1}) \oplus e_l J_{n-1}$ in $k\Sigma_l$-mod

   $x =$  $\rightsquigarrow x \in J_2 \subset J_4$

   $y =$ $\rightsquigarrow y \in J_4, y \notin J_2$

2. $e_l(J_n/J_{n-1}) \simeq ind_n(e_l(J_n/J_{n-1})e_n)$ as $k\Sigma_l$-modules
3. $res_n Ind_l M^\lambda$ admits a filtration by Specht modules for $k\Sigma_n$
4. $Res_l ind_n S_\nu$ admits a filtration by Specht modules for $k\Sigma_l$

Results

Let k, A be as above.

Theorem 1

$Y(l,\lambda)$ is the relative projective cover of $\mathrm{ind}_l S_\lambda$ with respect to the category of cell filtered modules.

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- $res_n Ind_l M^\lambda$ cell filtered $\rightsquigarrow$ $Ind_l M^\lambda$ cell filtered
- $char k \neq 2, 3 \Rightarrow \exists$ quasi-hereditary algebra $S(A)$ s.th. the category of cell filtered modules of A is equivalent to the category of standard filtered modules of $S(A)$ as exact categories [Hemmer-Nakano 2005; Dlab-Ringel 1992, HHKP 2010]

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- $Y^\lambda \twoheadrightarrow S_\lambda$

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- $ind_l Y^\lambda \twoheadrightarrow ind_l S_\lambda$

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- $Y(l,\lambda)$ cell filtered
- $Y(l,\lambda) \twoheadrightarrow ind_l Y^\lambda \twoheadrightarrow ind_l S_\lambda$
- Y^λ relative projective in the category of Specht filtered modules, $Res_l ind_n S_\lambda$ cell filtered $\rightsquigarrow$ $Y(l,\lambda)$ relative projective

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Theorem 2

There is a decomposition

$$Ind_l M^\lambda = \bigoplus_{(m\mu)\succeq(l,\lambda)} Y(m,\mu)^{a_{m,\mu}}$$

with non-negative integers $a_{m,\mu}$. Moreover, $a_{l,\lambda} = 1$.

Summary

- HP: Definition and properties of permutation modules for Brauer algebras, using explicit construction for the Brauer algebra
- HHKP: Structural properties, general definition of Young modules, but no explicit construction
- B:
  - Explicit construction of permutation modules/Young modules for cellularly stratified algebras satisfying the assumptions
  - Reprove HP's results using structural properties $\rightsquigarrow$ HP's results are valid in general