On the classification of naturally reductive homogeneous spaces in small dimensions

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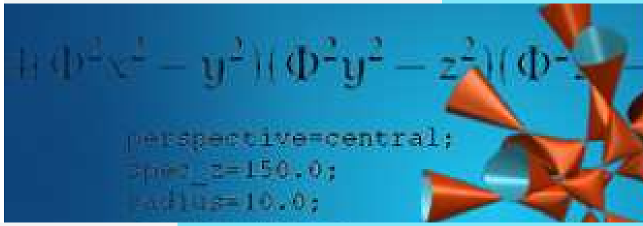

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– joint work with Ana Ferreira and Thomas Friedrich –

Naturally reductive homogeneous spaces

Traditional approach: (M, g) a Riemannian mnfld, $M = G/H$ s. t. G is a group of isometries acting transitively and effectively

Dfn. $M = G/H$ is *naturally reductive* if $\mathfrak{h}$ admits a reductive complement $\mathfrak{m}$ in $\mathfrak{g}$ s. t.

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle \;=\; 0 \;\text{ for all } X,Y,Z\in\mathfrak{m}, \qquad (*)$$

where $\langle -,-\rangle$ denotes the inner product on $\mathfrak{m}$ induced from g.
The PFB $G \to G/H$ induces a metric connection ∇ with torsion

$$T(X,Y,Z) := g(\nabla_X Y - \nabla_Y X - [X,Y], Z) = -\langle [X,Y]_{\mathfrak{m}}, Z\rangle,$$

the so-called *canonical connection*. It always satisfies $\nabla T = \nabla \mathcal{R} = 0.$

Observations:

- If G/H is symmetric, then $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, hence $T = 0$ and $\nabla =$ Levi-Civita connection ∇^g
- condition $(*)$ $\Leftrightarrow$ T is a 3-form, i. e. $T \in \Lambda^3(M)$.

Conversely:

Thm. A Riemannian manifold equipped with a [regular] homogeneous structure, i. e. a metric connection ∇ with torsion T and curvature $\mathcal{R}$ such that $\nabla\mathcal{R} = 0$ and $\nabla T = 0$, is locally isometric to a homogeneous space. [Ambrose-Singer, 1958, Tricerri 1993]

Hence: Naturally reductive spaces have a metric connection ∇ with skew torsion (T is 3-form) such that $\nabla T = \nabla\mathcal{R} = 0$

$\longrightarrow$ generalisation of Riemannian symm. spaces (class. by Cartan)

However, a classification in all dimensions is impossible!

Main pb: $\nexists$ invariant theory for $\Lambda^3(\mathbb{R}^n)$ under $\mathrm{SO}(n)$ for $n \geq 6$

- Use the recent progress on *metric connections with [parallel] skew torsion*
- Use *torsion* (instead of curvature) as basic geometric quantity, *find a G-structure* inducing the nat. red. structure

In this talk: General strategy, some general results, classification for $n \leq 6$

(Not in this talk: applications of the classification)

Set-up: (M,g) Riemannian mnfd, ∇ metric conn., ∇^g Levi-Civita conn.

$$T(X,Y,Z) \;=\; g(\nabla_X Y - \nabla_Y X - [X,Y], Z) \in \Lambda^3(M^n)$$

$$\nabla_X Y = \nabla^g_X Y + \tfrac{1}{2} T(X,Y,-)$$

(M,g,T) carries nat. red. homog. structure if $\nabla\mathcal{R} = 0$ and $\nabla T = 0$

Obviously: | nat.red.homog. Riemannian mnfds | $\subset$ | (homogeneous) Riemannian mnfds with parallel skew torsion |
|---|---|---|

N.B. Well-known: Some mnfds carry several nat.red.structures, for exa.

$$S^{2n+1} = \mathrm{SO}(2n+2)/\mathrm{SO}(2n+1) = \mathrm{SU}(n+1)/\mathrm{SU}(n),$$

$$S^6 \;=\; G_2/\mathrm{SU}(3),\; S^7 \;=\; \mathrm{Spin}(7)/G_2,\; S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7).$$

But: If (M,g) is not loc. isometric to a sphere or a Lie group, then its admits at most one naturally reductive homogeneous structure. [Olmos-Reggiani, 2012]

Review of some classical results

• all isotropy irreducible homogeneous manifolds are naturally reductive

• the $\pm$-connections on any Lie group with a biinvariant metric are naturally reductive (and, by the way, flat) [Cartan-Schouten, 1926]

• construction / classification (under some assumptions) of left-invariant naturally reductive metrics on compact Lie groups [D'Atri-Ziller, 1979]

• All 6-dim. homog. nearly Kähler mnfds (w. r. t. their canonical almost Hermitian structure) are naturally reductive. These are precisely: $S^3 \times S^3$, $\mathbb{CP}^3$, the flag manifold $F(1,2) = \mathrm{U}(3)/\mathrm{U}(1)^3$, and $S^6 = G_2/\mathrm{SU}(3)$.

• Known classifications:

- dimension 3 [Tricerri-Vanhecke, 1983], dimension 4 [Kowalski-Vanhecke, 1983], dimension 5 [Kowalski-Vanhecke, 1985]

These proceed by finding normal forms for the curvature operator, more details to follow later.

An important tool: the 4-form σ_T

Dfn. For any $T \in \Lambda^3(M)$, define ($e_1, \ldots, e_n$ a local ONF)

$$\sigma_T := \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T) \quad (= 0 \text{ if } n \leq 4)$$

[Exa: For $T = \alpha\, e_{123} + \beta\, e_{456}$, $\sigma_T = 0$; for $T = (e_{12} + e_{34})e_5$, $\sigma_T = -e_{1234}$]

- **σ_T measures the 'degeneracy' of T and, if non degenerate, induces the geometric structure on M**

- σ_T appears in many important relations:

  * 1st Bianchi identity: $\overset{X,Y,Z}{\mathfrak{S}}\ \mathcal{R}(X,Y,Z,V) = \sigma_T(X,Y,Z,V)$

  * $T^2 = -2\sigma_T + \|T\|^2$ in the Clifford algebra

  * If $\nabla T = 0$: $dT = 2\sigma_T$ and $\nabla^g T = \frac{1}{2}\sigma_T$

σ_T and the Nomizu construction

Idea: for $M = G/H$, reconstruct $\mathfrak{g}$ from $\mathfrak{h}$, T, $\mathcal{R}$ and $V \cong T_xM$

Set-up: $\mathfrak{h}$ a real Lie algebra, V a real f. d. $\mathfrak{h}$-module with $\mathfrak{h}$-invariant pos. def. scalar product $\langle , \rangle$, i. e. $\mathfrak{h} \subset \mathfrak{so}(V) \cong \Lambda^2 V$

$\mathcal{R} : \Lambda^2 V \to \mathfrak{h}$ an $\mathfrak{h}$-equivariant map, $T \in (\Lambda^3 V)^{\mathfrak{h}}$ an $\mathfrak{h}$-invariant 3-form,

Define a Lie algebra structure on $\mathfrak{g} := \mathfrak{h} \oplus V$ by $(A, B \in \mathfrak{h}, X, Y \in V)$:

$$[A + X, B + Y] := ([A, B]_{\mathfrak{h}} - \mathcal{R}(X, Y)) + (AY - BX - T(X, Y))$$

Jacobi identity for $\mathfrak{g}$ $\Leftrightarrow$

- $\overset{X,Y,Z}{\mathfrak{S}} \; \mathcal{R}(X, Y, Z, V) \; = \; \sigma_T(X, Y, Z, V)$ (1st Bianchi condition)
- $\overset{X,Y,Z}{\mathfrak{S}} \; \mathcal{R}(T(X, Y), Z) \; = \; 0$ (2nd Bianchi condition)

Observation: If (M, g, T) satisfies $\nabla T = 0$, then $\mathcal{R} : \Lambda^2(M) \to \Lambda^2(M)$ is symmetric (as in the Riemannian case).

Consider $\mathcal{C}(V) := \mathcal{C}(V, -\langle , \rangle)$: Clifford algebra, (recall: $T^2 = -2\sigma_T + \|T\|^2$)

Thm. If $\mathcal{R} : \ \Lambda^2 V \to \mathfrak{h} \subset \Lambda^2 V$ is symmetric, the first Bianchi condition is equivalent to $T^2 + \mathcal{R} \in \mathbb{R} \subset \mathcal{C}(V)$ $(\Leftrightarrow \ 2\sigma_T = \mathcal{R} \subset \mathcal{C}(V))$, and the second Bianchi condition holds automatically.

Exists in the literature in various formulations: based on an algebraic identity (Kostant); crucial step in a formula of Parthasarathy type for the square of the Dirac operator (A, '03); previously used by Schoemann 2007 and Fr. 2007, but without a clear statement nor a proof.

Practical relevance: allows to evaluate the 1st Bianchi identity in one condition!

Splitting theorems

Dfn. For T 3-form, define [introduced in AFr, 2004]

- kernel: $\ker T = \{X \in TM \,|\, X \lrcorner T = 0\}$
- Lie algebra generated by its image: $\mathfrak{g}_T := \mathrm{Lie}\langle X \lrcorner T \,|\, X \in V\rangle$

  $\mathfrak{g}_T$ is *not* related in any obvious way to the isotropy algebra of T!

Thm 1. Let (M, g, T) be a c. s. c. Riemannian mfld with parallel skew torsion T. Then $\ker T$ and $(\ker T)^{\perp}$ are ∇-parallel and ∇^g-parallel integrable distributions, M is a Riemannian product s. t.

$$(M, g, T) = (M_1, g_1, T_1 = 0) \times (M_2, g_2, T_2), \quad \ker T_2 = \{0\}$$

Thm 2. Let (M, g, T) be a c. s. c. Riemannian mfld with parallel skew torsion T s. t. $\sigma_T = 0$, $TM = \mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_q$ the decomposition of TM in $\mathfrak{g}_T$-irreducible, ∇-par. distributions. Then all $\mathcal{T}_i$ are ∇^g-par. and integrable, M is a Riemannian product, and the torsion T splits accordingly

$$(M, g, T) = (M_1, g_1, T_1) \times \ldots \times (M_q, g_q, T_q)$$

A structure theorem for vanishing σ_T

Thm. Let (M^n, g) be an *irreducible*, c. s. c. Riemannian mnfld with parallel skew torsion $T \neq 0$ s. t. $\sigma_T = 0$, $n \geq 5$. Then M^n is a simple compact Lie group with biinvariant metric or its dual noncompact symmetric space.

Key ideas: $\sigma_T = 0 \Rightarrow$ Nomizu construction yields Lie algebra structure on TM

use $\mathfrak{g}_T$; use a Skew Holonomy Theorem by Olmos-Reggiani (2012), based on A-Fr (2004), to show that G_T is simple and acts on TM by its adjoint rep.

prove that $\mathfrak{g}_T = \mathfrak{iso}(T) = \mathfrak{hol}^g$, hence acts irreducibly on TM, hence M is an irred. symmetric space by Berger's Thm

Exa. Fix $T \in \Lambda^3(\mathbb{R}^n)$ with constant coefficients s. t. $\sigma_T = 0$. Then the flat space $(\mathbb{R}^n, g, T)$ is a reducible Riemannian mnfld with parallel skew torsion and $\sigma_T = 0 \to$ assumption 'M irreducible' is crucial! (the Riemannian manifold is decomposable, but the torsion is not)

Classification of nat. red. spaces in $n = 3$

[Tricerri-Vanhecke, 1983]

Then $\sigma_T = 0$, and the Nomizu construction can be applied directly to obtain in a few lines:

Thm. Let $(M^3, g, T \neq 0)$ be a 3-dim. c. s. c. Riemannian mnfld with a naturally reductive structure. Then (M^3, g) is one of the following:

- $\mathbb{R}^3, S^3$ or $\mathbb{H}^3$;
- isometric to one of the following Lie groups with a suitable left-invariant metric:

$$SU(2), \quad \widetilde{SL}(2,\mathbb{R}), \quad \text{or the 3-dim. Heisenberg group } H^3$$

N.B. A general classification of mnfds with par. skew torsion is meaninless – any 3-dim. volume form of a metric connection is parallel.

Proof: $T = \lambda\, e_{123}$; M is either Einstein ($\to$ space form) or $\mathfrak{hol}^{\nabla}$ is one-dim., i. e. $\mathfrak{hol}^{\nabla} \;=\; \mathbb{R}\cdot\Omega$ and $\mathcal{R} \;=\; \alpha\,\Omega\odot\Omega$.

By the Nomizu construction, $e_1, e_2, e_3,$ and Ω are a basis of $\mathfrak{g}$ with commutator relations

$$[e_1, e_2] = -\alpha\Omega - \lambda e_3 =: \tilde{\Omega}, \quad [e_1, e_3] = \lambda e_2, \quad [e_2, e_3] = -\lambda e_1,$$
$$[\Omega, e_1] = e_2, \quad [\Omega, e_2] = -e_1, \quad [\Omega, e_3] = 0.$$

The 3-dimensional subspace $\mathfrak{h}$ spanned by e_1, e_2, and $\tilde{\Omega}$ is a Lie subalgebra of $\mathfrak{g}$ that is transversal to the isotropy algebra $\mathfrak{k}$ (since $\lambda \neq 0$). Consequently, M^3 is a Lie group with a left invariant metric. One checks that $\mathfrak{h}$ has the commutator relations

$$[e_1, e_2] = \tilde{\Omega}, \quad [\tilde{\Omega}, e_1] = (\lambda^2 - \alpha)e_2, \quad [e_2, \tilde{\Omega}] = (\lambda^2 - \alpha)e_1.$$

For $\alpha = \lambda^2$, this is the 3-dimensional Heisenberg Lie algebra, otherwise it is $\mathfrak{su}(2)$ or $\mathfrak{sl}(2,\mathbb{R})$ depending on the sign of $\lambda^2 - \alpha$.

Classification of nat. red. spaces in $n = 4$

Thm. $(M^4, g, T \neq 0)$ a c. s. c. Riem. 4-mnfld with parallel skew torsion. Then

1) $V := *T$ is a ∇^g-parallel vector field.

2) $\mathrm{Hol}(\nabla^g) \subset \mathrm{SO}(3)$, hence M^4 is isometric to a product $N^3 \times \mathbb{R}$, where (N^3, g) is a 3-manifold with a parallel 3-form T.

• T has normal form $T = e_{123}$, so $\dim\ker T = 1$ and 2) follows at once from our 1st splitting thm: but the existence of V explains directly & *geometrically* the result in a few lines.

• Thm shows that the next result does not rely on the curvature or the homogeneity

Since a R. product is is nat. red. iff both factors are nar. red., we conclude:

Cor. A 4-dim. naturally reductive Riemannian manifold with $T \neq 0$ is locally isometric to a Riemannian product $N^3 \times \mathbb{R}$, where N^3 is a 3-dimensional naturally reductive Riemannian manifold. [Kowalski-Vanhecke, 1983]

Classification of nat. red. spaces in $n = 5$

Assume $(M^5, g, T \neq 0)$ is Riemannian mnfd with parallel skew torsion

- $\exists$ a local frame s. t (for constants $\lambda, \varrho \in \mathbb{R}$)

$$T = -(\varrho e_{125} + \lambda e_{345}), \quad *T = -(\varrho e_{34} + \lambda e_{12}), \quad \sigma_T = \varrho\lambda e_{1234}$$

- **Case A:** $\sigma_T = 0$ ($\Leftrightarrow \varrho\lambda = 0$): apply 2nd splitting thm, M^5 is then loc. a product $N^3 \times N^2$ (if nat. red., N^2 has constant Gaussian curvature)

- **Case B:** $\sigma_T \neq 0$, two subcases:

  * Case B.1: $\lambda \neq \varrho$, $\mathrm{Iso}(T) = \mathrm{SO}(2) \times \mathrm{SO}(2)$

  * Case B.2: $\lambda = \varrho$, $\mathrm{Iso}(T) = \mathrm{U}(2)$

Recall: Given a G-structure on (M, g), a *characteristic connection* is a metric connection with skew torsion preserving the G-structure (if existent, it's unique)

$n = 5$: The induced contact structure

Case B: $\sigma_T \neq 0$

Dfn. A metric almost contact structure (φ, η) on (M^{2n+1}, g) is called (N: Nijenhuis tensor, $F(X,Y) := g(X, \varphi Y)$)

- quasi-Sasakian if $N = 0$ and $dF = 0$
- α-Sasakian if $N = 0$ and $d\eta = \alpha F$ (Sasaki: $\alpha = 2$)

Thm. Let (M^5, g, T) be a Riemannian 5-mnfld with parallel skew torsion T such that $\sigma_T \neq 0$. Then M is a quasi-Sasakian manifold and ∇ is its characteristic connection.

The structure is α-Sasakian iff $\lambda = \varrho$ (case B.2), and it is Sasakian if $\lambda = \varrho = 2$.

Construction: $V := *\sigma_T \neq 0$ is a ∇-parallel Killing vector field of constant length

$\equiv$ contact direction $\eta = e_5$ (up to normalisation)

Check: $T = \eta \wedge d\eta$, define $F = -(e_{12} + e_{34})$, then prove that this works.

$n = 5$: Classification I

For $\lambda = \varrho$ (case B.2), no classification for parallel skew torsion is possible (many non-homogeneous Sasakian mnfds are known). But for

Case B.1: $\lambda \neq \varrho$

Thm. Let (M^5, g, T) be Riemannian 5-manifold with parallel skew torsion s. t. T has the normal form

$$T = -(\varrho e_{125} + \lambda e_{345}), \quad \varrho\lambda \neq 0 \text{ and } \varrho \neq \lambda.$$

Then $\nabla\mathcal{R} = 0$, i. e. M is locally naturally reductive, and the family of admissible torsion forms and curvature operators depends on 4 parameters.

[Use Clifford criterion to relate $\mathcal{R}$ and σ_T]

Now one can apply the Nomizu construction to obtain the classification:

$n = 5$: Classification II

Thm. A c. s. c. Riemannian 5-mnfld (M^5, g, T) with parallel skew torsion $T = -(\varrho e_{125} + \lambda e_{345})$ with $\varrho\lambda \neq 0$ is isometric to one of the following naturally reductive homogeneous spaces:

If $\lambda \neq \varrho$ (B.1):

a) The 5-dimensional Heisenberg group H^5 with a two-parameter family of left-invariant metrics,

b) A manifold of type $(G_1 \times G_2)/\mathrm{SO}(2)$ where G_1 and G_2 are either $\mathrm{SU}(2)$, $\mathrm{SL}(2, \mathbb{R})$, or H^3, but not both equal to H^3 with one parameter $r \in \mathbb{Q}$ classifying the embedding of $\mathrm{SO}(2)$ and a two-parameter family of homogeneous metrics.

If $\lambda = \varrho$ (B.2): One of the spaces above or $\mathrm{SU}(3)/\mathrm{SU}(2)$ or $\mathrm{SU}(2,1)/\mathrm{SU}(2)$ (the family of metrics depends on two parameters).

[Kowalski-Vanhecke, 1985]

Example: The $(2n+1)$-dimensional Heisenberg group

$$H^{2n+1} = \left\{ \begin{bmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} ; x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \cong \mathbb{R}^{2n+1},$$ local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$

- Metric: described by parameters $\lambda = (\lambda_1, \ldots, \lambda_n)$, all $\lambda_i > 0$

$$g_\lambda = \sum_{i=1}^{n} \frac{1}{\lambda_i}(dx_i^2 + dy_i^2) + \left[dz - \sum_{j=1}^{n} x_j dy_j \right]^2$$

- Contact str.: $\eta = dz - \sum_{i=1}^{n} x_i dy_i$, $F = -\sum_{i=1}^{n} \frac{1}{\lambda_i} dx_i \wedge dy_i$

- Characteristic connection ∇: torsion: $T = \eta \wedge d\eta = -\sum_{i=1}^{n} \eta \wedge dx_i \wedge dy_i$

- Curvature: $\mathcal{R} = \sum_{i \leq j}^{n} \sqrt{\lambda_i \lambda_j} (dx_i \wedge dy_i)^2$ [read as symm. tensor product of 2-forms]

Now check that $\nabla T = \nabla \mathcal{R} = 0$.

The case $n = 6$ I

Assume $\ker T = 0$ from beginning. Distinction $\sigma_T =, \neq 0$ is too crude.

$*\sigma_T$: a 2-form $\equiv$ skew-symm. endomorphism, classify by its **rank!** (=0,2,4,6 / Case A, B, C, D)

Geometry: Can $*\sigma_T$ be interpreted as an almost complex structure?

Recall: $\Lambda^3(\mathbb{R}^6) \stackrel{\mathrm{SU}(3)}{=} W_1^{(2)} \oplus W_3^{(12)} \oplus W_4^{(6)}$:
type of torsion $T \in \Lambda^3(\mathbb{R}^6)$ describes all almost hermitian mnfds with characteristic connection [Gray-Hervella, 1980; Friedrich-Ivanov, 2003]

Exa. On $S^3 \times S^3$, there exist 3-forms with the following subcases:

Type	$W_1 \oplus W_3$	W_1	$W_3 \oplus W_4$	$--$
$\mathrm{rk}\,(*\sigma_T)$	6	6	2	0
$\mathfrak{iso}(T)$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	T^2	$\mathfrak{so}(3) \times \mathfrak{so}(3)$

Case A: $\sigma_T = 0$

This covers, for example, torsions of form $\mu\, e_{123} + \nu\, e_{456}$. This is basically all by our 2nd splitting thm:

Thm. A c. s. c. Riemannian 6-mnfld with parallel skew torsion T s. t. $\sigma_T = 0$ and $\ker T = 0$ splits into two 3-dimensional manifolds with parallel skew torsion,

$$(M^6, g, T) = (N_1^3, g_1, T_1) \times (N_2^3, g_2, T_2)$$

Cor. Any 6-dim. nat. red. homog. space with $\sigma_T = 0$ and $\ker T = 0$ is locally isometric to a product of two 3-dimensional nat. red. homog. spaces.

The case $n = 6$ II

Case B: $\mathrm{rk}\,(*\sigma_T) = 2$

A priori, it is not possible to define an almost complex structure.

Thm. Let (M^6, g, T) be a 6-mnfd with parallel skew torsion s. t. $\ker T = 0$, $\mathrm{rk}\,(*\sigma_T) = 2$. Then $\nabla\mathcal{R} = 0$, i. e. M is nat. red., and there exist constants $a, b, c, \alpha, \beta \in \mathbb{R}$ s. t.

$$T = \alpha(e_{12} + e_{34}) \wedge e_5 + \beta(e_{12} - e_{34}) \wedge e_6$$

$$\mathcal{R} = a(e_{12} + e_{34})^2 + c(e_{12} + e_{34}) \odot (e_{12} - e_{34}) + b(e_{12} - e_{34})^2$$

with the relation $a + b = -(\alpha^2 + \beta^2)$.

Now perform Nomizu construction to conclude:

Thm. A c. s. c. Riemannian 6-mnfd with parallel skew torsion T and $\mathrm{rk}\,(*\sigma_T) = 2$ is the product $G_1 \times G_2$ of two Lie groups equipped with a family of left invariant metrics. G_1 and G_2 are either $S^3 = \mathrm{SU}(2)$, $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, or H^3.

The case $n = 6$ III

Case B: $\mathrm{rk}\,(*\sigma_T) = 4$

Thm. For the torsion form of a metric connection with parallel skew torsion ($\ker T = 0$), the case $\mathrm{rk}\,(*\sigma_T) = 4$ cannot occur.

[but: such forms exist if $\nabla T \neq 0$! – these results explain why a classification is possible without knowing the orbit class. of $\Lambda^3(\mathbb{R}^6)$ under $\mathrm{SO}(6)$]

The case $n = 6$ IV

Case C: $\mathrm{rk}\,(*\sigma_T) = 6$

Thm. Such a 6-mnfd with parallel skew torsion admits an almost complex structure J of Gray-Hervella class $W_1 \oplus W_3$.

All three eigenvalues of $*\sigma_T$ are equal, hence $*\sigma_T$ is proportional to Ω, the fundamental form of J. It's either nearly Kähler (W_1), or it is naturally reductive and $\mathfrak{hol}^\nabla = \mathfrak{so}(3)$.

N.B. If class W_1 (M^6 nearly Kähler mnfd): the only homogeneous ones are $S^6, S^3 \times S^3, \mathbb{CP}^3, F(1,2)$. [Butruille, 2005]

It is not known whether there exist non-homogeneous nearly Kähler mnfds.

Again, we have an explicit formula for torsion and curvature, then perform the Nomizu construction (. . . and survive).

The case $n = 6$ V

Final result of Nomizu construction:

Thm. A c. s. c. Riemannian 6-mnfd with parallel skew torsion T, $\mathrm{rk}\,(*\sigma_T) = 6$ and $\ker T = 0$ that is *not* isometric to a nearly Kähler manifold is one of the following Lie groups with a suitable family of left-invariant metrics:

- The nilpotent Lie group with Lie algebra $\mathbb{R}^3 \times \mathbb{R}^3$ with commutator $[(v_1, w_1), (v_2, w_2)] = (0, v_1 \times v_2)$,
- the direct or the semidirect product of S^3 with $\mathbb{R}^3$,
- the product $S^3 \times S^3$,
- the Lie group $\mathrm{SL}(2, \mathbb{C})$, viewed as a real mnfld (with a deformed complex str.!)

- prove that manifold is indeed a Lie group,
- identify its abstract Lie algebra by degeneracy / EV of its Killing form,
- find 3-dim. subalgebra defining a 3-dim. quotient and prove that the 6-dim. Lie alg. is its isometry algebra;

for example, $\mathrm{SL}(2, \mathbb{C})$ appears because it's the isometry group of hyperbolic space $\mathbb{H}^3$

Homework. Identify the 6-dimensional Lie algebra $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{h} = \mathrm{span}(\Omega_1, \Omega_3, \Omega_5), \mathfrak{m} := \mathrm{span}(e_2, e_4, e_6)$ defined by $(\alpha, \alpha', \beta \in \mathbb{R})$

$$[\Omega_1, \Omega_3] = (\alpha - 2\beta)\Omega_5, \quad [\Omega_1, \Omega_5] = (2\beta - \alpha)\Omega_3, \quad [\Omega_3, \Omega_5] = (\alpha - 2\beta)\Omega_1$$

$$[\Omega_1, e_4] = [e_2, \Omega_3] = (\alpha - 2\beta)e_6, \ [\Omega_1, e_6] = [e_2, \Omega_5] = (2\beta - \alpha)e_4,$$

$$[\Omega_3, e_6] = [e_4, \Omega_5] = (\alpha - 2\beta)e_2.$$

$$[e_2, e_4] = -\beta\Omega_5 - \alpha' e_6, \quad [e_2, e_6] = \beta\Omega_3 + \alpha' e_4, \quad [e_4, e_6] = -\beta\Omega_1 - \alpha' e_2.$$

and use it to deduce the previous theorem.

Hint: Prove first that $\mathfrak{g}$ is not semisimple iff $\alpha = 2\beta$ or $4\beta(\alpha - 2\beta) = {\alpha'}^2$.

Example: $\mathrm{SL}(2,\mathbb{C})$ viewed as a 6-dimensional real mnfd

- Write $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus i\,\mathfrak{su}(2)$;

Killing form $\beta(X,Y)$ is neg. def. on $\mathfrak{su}(2)$, pos. def.on $i\,\mathfrak{su}(2)$

- $M^6 = G/H = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(2)/\mathrm{SU}(2)$ with $H = \mathrm{SU}(2)$ embedded diag

(recall that $\mathfrak{hol}^\nabla = \mathfrak{so}(3)$; want that isotropy rep. = holonomy rep.)

- $\mathfrak{m}_\alpha$ red. compl. of $\mathfrak{h}$ inside $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{su}(2)$ depending on $\alpha \in \mathbb{R} - \{1\}$,

$$\mathfrak{h} \ = \ \{(B,B) \ : \ B \in \mathfrak{su}(2)\}, \quad \mathfrak{m}_\alpha \ := \ \{(A+\alpha B, B) \ : \ A \in i\,\mathfrak{su}(2),\ B \in \mathfrak{su}(2)\}.$$

- Riemannian metric:

$$g_\lambda((A_1+\alpha B_1, B_1),(A_2+\alpha B_2, B_2)) := \beta(A_1,A_2) - \tfrac{1}{\lambda^2}\beta(B_1,B_2), \quad \lambda > 0$$

- In suitable ONB: almost hermitian str.: $\Omega := x_{12} + x_{34} + x_{56}$ with torsion

$$T = N + d\Omega \circ J = \left[2\lambda(1-\alpha) + \tfrac{4}{\lambda(1-\alpha)}\right] x_{135} + \tfrac{2}{\lambda(1-\alpha)}[x_{146} + x_{236} + x_{245}].$$

- Curvature: has to be a map $\mathcal{R} : \Lambda^2(M^6) \to \mathfrak{hol}^\nabla \subset \mathfrak{so}(6)$, here: mainly projection on $\mathfrak{hol}^\nabla = \mathfrak{so}(3)$.

- $\nabla T = \nabla \mathcal{R} = 0$, i. e. naturally reductive for all α, λ; type $W_1 \oplus W_3$ or W_3

The skew torsion holonomy theorem

Dfn. Let $0 \neq T \in \Lambda^3(V)$, $\mathfrak{g}_T$ as before, $G_T \subset \mathrm{SO}(n)$ its Lie group. Hence, $X \lrcorner T \in \mathfrak{g}_T \subset \mathfrak{so}(V) \cong \Lambda^2(V)\ \forall\ X \in V$. Then (G_T, V, T) is called a *skew-torsion holonomy system (STHS)*. It is said to be

- *irreducible* if G_T acts irreducibly on V,
- *transitive* if G_T acts transitively on the unit sphere of V,
- and *symmetric* if T is G_T-invariant.

Recall: The only transitive sphere actions are:

$\mathrm{SO}(n)$ on $S^{n-1} \subset \mathbb{R}^n$, $\mathrm{SU}(n)$ on $S^{2n-1} \subset \mathbb{C}^n$, $\mathrm{Sp}(n)$ on $S^{4n-1} \subset \mathbb{H}^n$, G_2 on S^6, $\mathrm{Spin}(7)$ on S^7, $\mathrm{Spin}(9)$ on S^{15}. [Montgomery-Samelson, 1943]

Thm (STHT). Let (G_T, V, T) be an irreducible STHS. If it is transitive, $G_T = \mathrm{SO}(n)$. If it is not transitive, it is symmetric, and

- V is a simple Lie algebra of rank ≥ 2 w. r. t. the bracket $[X, Y] = T(X, Y)$, and G_T acts on V by its adjoint representation,
- T is unique up to a scalar multiple.

[transitive: AFr 2004, general: Olmos-Reggiani, 2012; Nagy 2013]